Vector Calculus on Weighted Networks

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Abstract

We present here a vector calculus on weighted networks following the guidelines of Differential Geometry. The key to develop an efficient calculus on weighted networks which mimetizes the calculus in the smooth case is an adequate construction of the tangent space at each vertex. This allows to consider discrete vector fields, inner products and general metrics. Then, we obtain discrete versions of derivative, gradient, divergence, curl and Laplace-Beltrami operators, satisfying analogous properties to those verified by their continuous counterparts. Also we construct the De Rham cohomology of a weighted networks, obtaining in particular a Hodge decomposition theorem type. On the other hand we develop the corresponding integral calculus that includes the discrete versions of the Integration by Parts technique and Green's Identities. As an application we study the variational formulation for general boundary value problems on weighted networks, obtaining in particular the discrete version of the Dirichlet Principle.

Key Words: Weighted networks, Vector Calculus, Discrete operators, Network Cohomology, Discrete Green's Identities, Discrete Boundary Value Problems.

1 Introduction

The discrete vector calculus theory is a very fruitful area of work in many mathematical branches not only for its intrinsic interest but also for its applications, [1, 4, 6, 9, 15, 17, 19, 21, 22]. One can construct a discrete vector calculus by considering simplicial complexes that approximates locally a smooth manifold and then use the Whitney application to define inner products on the cochain spaces. This gives rise to a combinatorial Hodge theory, allows to translate the basic notions of Riemannian geometry into combinatorial terms and shows that the combinatorial objects are good approximations for the smooth ones, [15].

Alternatively, one can approximate a smooth manifold by means of non-simplicial meshes and then define discrete operators either by truncating the smooth ones or interpolating on the mesh elements. This approach is considered in the aim of mimetic methods which are used in the context of difference schemes to solve numerically boundary values problems. These methods have good computational properties, [16, 17].

Another approach is to deal with the mesh as the unique existent space and then the discrete vector calculus is described throughout tools from the Algebraic Topology since the geometric realization of the mesh is a unidimensional CW-complex, [1, 19, 22]. The discrete operators can be defined in combinatorial terms and then the main tool is the incidence matrix associated with an oriented graph, [7, 8].

Our work falls within the last ambit but, instead of importing the tools from Algebraic Topology, we construct the discrete vector calculus from the graph structure itself following the guidelines of Differential Geometry. The key to develop our discrete calculus is an adequate construction of the tangent space at each vertex of the graph. The concepts of discrete vector fields and bilinear forms are a likely result of the definition of tangent space. Moreover, they are general, while only orthogonal bilinear forms and vector fields that are either symmetric or antisymmetric are habitually considered in the literature. We obtain discrete versions of the derivative, gradient, divergence, curl and Laplace-Beltrami operators that satisfy the same properties that its continuum analogues. We also introduce the notion of order of an operator that recognizes the Laplace-Beltrami operator as a second order operator, while the rest of the above-mentioned operators are of first order. Moreover, we construct the De Rham cohomology of a weighted network, obtaining in particular discrete analogues of the Poincaré and the Hodge decomposition theorems.

Unlike other works, here it is not necessary to provide the weighted network with an orientation to develop a satisfactory discrete vector calculus. However, we consider both the oriented version and the unoriented one taking advantage of both approaches. We must note that the Laplace-Beltrami operator does not depend on the chosen orientation for orthogonal metrics, whereas this does not happen for general metrics.

We also develop an integral calculus that includes the discrete versions of the integration along curves, Integration by Parts formulae and the Green's Identities. As a consequence we describe appropriately general boundary value problems on arbitrary nonempty subsets of weighted networks as well as its variational formulation. Then, we obtain necessary and sufficient condition for the existence and uniqueness of solution. Moreover, we prove a discrete version of the Dirichlet Principle for self-adjoint boundary value problems associated with elliptic operators.

2 Preliminaries

Along the paper, $\Gamma = (V, E)$ will denote a simple and finite connected graph without loops, with vertex set V and edge set E, although almost all concepts can be extended to infinite and locally finite graphs. The number $\chi(\Gamma) = |V| - |E|$ is called the *Euler characteristic of* Γ. It is well-known that $\chi(\Gamma) \leq 1$ and the equality is verified iff Γ is a tree.

Two different vertices, $x, y \in V$, are called *adjacent*, which will be represented by $x \sim y$, if $\{x, y\} \in E$. In this case, the edge $\{x, y\}$ will be represented as e_{xy} and the vertices x and y are called incidents with e_{xy}. In addition, for any $x \in V$ the value $k(x)$ will denote the number of vertices adjacent to x.

An *orientation on* Γ is an application $\tau\colon E \longrightarrow V$ such that for all $e \in E$, $\tau(e)$ is incident with e. The vertex $\tau(e)$ will be called *head of* e, whereas the vertex $\zeta(e) \in V$ such that $e = \{\tau(e), \zeta(e)\}$ will be called *tail of* e.

If $x, y \in V$, a *curve of length* n from x to y is an ordered sequence of $n + 1$ vertices, $\alpha = \{x_0, \dots, x_n\}$, such that $x_0 = x$, $x_n = y$ and $x_j \sim x_{j+1}$, $j = 0, \dots, n - 1$. In this case x and y are called the *ends of the curve*. A *closed curve* is a curve whose ends coincide. For any $x, y \in V$, the length of the shortest curve joining x and y will be denoted by $d(x,y)$ and it is well-known that d defines a distance on the graph.

We will denote by $\mathcal{C}(V)$, $\mathcal{C}(E)$, $\mathcal{C}(V \times V)$ and $\mathcal{C}(V \times V \times V)$, the vector spaces of real functions defined on the sets that appear between brackets. Moreover, if $u \in \mathcal{C}(V)$, the *support of* u is the set $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$.

To develop a difference calculus that follows the guidelines of the differential one it will be crucial to define functions that contemplate the relation between vertices and its incident edges. For this, we will consider the following subsets:

$$\mathcal{C}(\Gamma) = \{f \in \mathcal{C}(V \times V) : f(x,y) = 0, \;\text{ if }\; d(x,y) \neq 1\},$$

$$\mathcal{C}(\Gamma \times \Gamma) = \{f \in \mathcal{C}(V \times V \times V) : f(x,y,z) = 0, \;\text{ if }\; d(x,y) \cdot d(x,z) \neq 1\}.$$

If $f \in \mathcal{C}(\Gamma)$, then f is called *symmetric* iff $f(x,y) = f(y,x)$, for each $x, y \in V$ or *antisymmetric* iff $f(x,y) = -f(y,x))$, for each $x, y \in V$. In addition, for each $f \in \mathcal{C}(\Gamma)$ we obtain that $f = f^s + f^a$, where $f^s(x,y) = \frac{1}{2}\left(f(x,y) + f(y,x)\right)$ and $f^a(x,y) = \frac{1}{2}\left(f(x,y) - f(y,x)\right)$. Moreover, as any $u \in \mathcal{C}(V)$ can be considered as a function of $\mathcal{C}(\Gamma)$ by defining $u(x,y) = u(x)$ for any $y \sim x$, the above decomposition has also sense for functions in $\mathcal{C}(V)$. On the other hand it is clear that the space $\mathcal{C}(E)$ is naturally identify with the subspace of $\mathcal{C}(\Gamma)$ formed by all symmetric functions.

Next we define the tangent space at a vertex of a graph. Unlike the continuous case, the dimension of the tangent space in a graph varies with each vertex. Since in topological terms a graph is unidimensional, one could presuppose that the tangent space should be also unidimensional. Such a though is ineffective in providing suitable information about the graph, because each vertex is connected with its adjacent vertices and these connections must be reflected as degrees of freedom in the tangent space. Definitely, since the tangent space at a vertex must distinguish the different ways of reaching or leaving the vertex, its dimension must coincide with the vertex degree. So, for each vertex $x \in V$, we will call the real vector space of formal linear combinations of the edges incident with x *tangent space at* x and we denote it by $T_x(\Gamma)$. It is clear that for any $x \in V$, the set of edges incident with x is a basis of $T_x(\Gamma)$, that will be called *coordinate basis of* $T_x(\Gamma)$. Therefore, $\dim T_x(\Gamma) = k(x)$.

We will call any application $\mathsf{f} : V \longrightarrow \bigcup\limits_{x \in V} T_x(\Gamma)$ such that $\mathsf{f}(x) \in T_x(\Gamma)$ for each $x \in V$ *vector field on* Γ. The *support of* f is defined as the set $\mathsf{supp}(\mathsf{f}) = \{x \in V : \mathsf{f}(x) \neq \mathsf{0}\}$.

If f is a vector field on Γ, then f is uniquely determined by its components in the coordinate basis. Therefore, we can associate with f the function $f \in \mathcal{C}(\Gamma)$ such that for each $x \in V$, $\mathsf{f}(x) = \sum\limits_{y \sim x} f(x,y)\, e_{xy}$. A vector field will be called *symmetric or antisymmetric* when its corresponding component function has the same property. If f is a vector field and $f \in \mathcal{C}(\Gamma)$ is its component function, the vector fields f^s and f^a whose component functions are f^s and f^a are called *symmetric part and antisymmetric part of* f, respectively. In what follows $\mathsf{1}$ will denote the symmetric vector field whose component function is given by $f(x,y) = 1$ when $x \sim y$ and $f(x,y) = 0$, otherwise.

The space of vector fields on Γ will be denoted by $\mathcal{X}(\Gamma)$, whereas the subspaces of symmetric or antisymmetric vector fields will be denoted by $\mathcal{X}^s(\Gamma)$ and $\mathcal{X}^a(\Gamma)$, respectively.

For each $x \in V$, let $\mathcal{T}^1_x(\Gamma)$ and $\mathcal{T}^2_x(\Gamma)$ be the vector spaces of endomorphisms and bilinear forms on $T_x(\Gamma)$, respectively. We will call any application $\mathsf{M}\colon V \longrightarrow \bigcup_{x\in V} \mathcal{T}^1_x(\Gamma)$ such that for any $x \in V$, $\mathsf{M}(x) \in \mathcal{T}^1_x(\Gamma)$ *field of endomorphisms on* Γ. The vector space of fields of endomorphisms on Γ will be denoted by $\mathcal{T}^1(\Gamma)$.

We will call any application $\mathsf{B}\colon V \longrightarrow \bigcup_{x\in V} \mathcal{T}^2_x(\Gamma)$ such that for any $x \in V$, $\mathsf{B}(x) \in \mathcal{T}^2_x(\Gamma)$ *field of bilinear forms on* Γ. The vector space of fields of bilinear forms on Γ will be denoted by $\mathcal{T}^2(\Gamma)$.

We will say that the field B of bilinear forms is *symmetric, non degenerated or orthogonal* if for any $x \in V$, $\mathsf{B}(x)$ is non degenerated, symmetric or the coordinate basis of $T_x(\Gamma)$ is orthogonal with respect to $\mathsf{B}(x)$. If for any $x \in V$, $\mathsf{B}(x)$ is an inner product on $T_x(\Gamma)$, then B will be called *field of inner products or metric on* Γ. In particular, we will call the unique metric on Γ such that for any $x \in V$, the coordinate basis of $T_x(\Gamma)$ is orthonormal *canonical metric on* Γ, which will be represented by $\langle\cdot,\cdot\rangle$. So, if $f, g \in \mathcal{C}(\Gamma)$ are the component functions of the vector fields $\mathsf{f}, \mathsf{g} \in \mathcal{X}(\Gamma)$, then $\langle \mathsf{f}, \mathsf{g}\rangle(x) = \sum_{y\in V} f(x,y)g(x,y)$, for any $x \in V$.

If $\mathsf{B} \in \mathcal{T}^2(\Gamma)$, we can consider the function $m \in \mathcal{C}(\Gamma \times \Gamma)$ such that if $x \in V$ and y, z are adjacent to x then $m(x,y,z) = \mathsf{B}(x)(e_{xy}, e_{xz})$. Clearly, B is uniquely determined by m, which will be called *component function of* B and so we can identify the vector space $\mathcal{T}^2(\Gamma)$ with $\mathcal{C}(\Gamma \times \Gamma)$. Then, the subspace of orthogonal bilinear form is identified with $\mathcal{C}(\Gamma)$.

The existence of a coordinate basis on $T_x(\Gamma)$ allows to identify naturally $\mathcal{T}^1(\Gamma)$ and $\mathcal{T}^2(\Gamma)$. If $\mathsf{B} \in \mathcal{T}^2(\Gamma)$ and $\mathsf{M} \in \mathcal{T}^1(\Gamma)$ are identified, the component function of B will also be called the *component function of* M. Therefore, a field of endomorphisms will be called *symmetric or diagonal* if its associated field of bilinear form is symmetric or orthogonal, respectively. In particular, if τ is an orientation on Γ, we will denote by D_τ the diagonal field of endomorphisms whose component function is given by

$$\mathsf{1}_\tau(x,y,z) = \begin{cases} 1, & \text{when } z = y \text{ and } \tau(e_{xy}) = y, \\ -1, & \text{when } z = y \text{ and } \tau(e_{xy}) = x, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, we can identify the space of vector fields $\mathcal{X}(\Gamma)$ with the subspace of $\mathcal{T}^1(\Gamma)$ formed by the diagonal fields of endomorphisms: If $\mathsf{f} \in \mathcal{X}(\Gamma)$ and $f \in \mathcal{C}(\Gamma)$ is its component function, we define $\mathsf{D}_\mathsf{f} \in \mathcal{T}^1(\Gamma)$ as the field of endomorphisms whose component function is given by $m(x,y,y) = f(x,y)$ and by $m(x,y,z) = 0$, otherwise.

If B is non degenerated and M is its associated field of endomorphisms, then $\mathsf{M}(x)$ is an automorphism for each $x \in V$. In this case, we denote by M^{-1} and by B^{-1} the fields of endomorphisms and bilinear forms determined by $\mathsf{M}^{-1}(x)$ for each $x \in V$. It is verified that B is orthogonal, symmetric or a metric iff B^{-1} also is.

If $\mathsf{M} \in \mathcal{T}^1(\Gamma)$ and $\mathsf{f} \in \mathcal{X}(\Gamma)$, the application $\mathsf{M}\mathsf{f}\colon V \longrightarrow T(\Gamma)$ that assigns to any $x \in V$

the element of $T_x(\Gamma)$ given by $\mathsf{Mf}(x) = \mathsf{M}(x)(\mathsf{f}(x))$, defines a vector field on Γ whose component function is given by $\sum\limits_{z\in V} m(x,y,z)f(x,z)$, for any $x,y \in V$ and therefore verifies that $\mathsf{supp}(\mathsf{Mf}) \subset \mathsf{supp}(\mathsf{f})$. Analogously, if $\mathsf{B} \in \mathcal{T}^2(\Gamma)$ has M as its associated field of endomorphisms and $\mathsf{f},\mathsf{g} \in \mathcal{X}(\Gamma)$, we can consider $\mathsf{B}(\mathsf{f},\mathsf{g}) \in \mathcal{C}(V)$, the application given by $\mathsf{B}(\mathsf{f},\mathsf{g})(x) = \langle \mathsf{Mf},\mathsf{g}\rangle(x)$, for each $x \in V$.

Let $\mathcal{V}_1 \subset \mathcal{X}(\Gamma)$, $\mathcal{V}_2 \subset \mathcal{C}(V)$ be vector subspaces and consider the linear applications

$$\mathsf{F}\colon \mathcal{V}_1 \longrightarrow \mathcal{X}(\Gamma),\quad \mathsf{G}: \mathcal{V}_2 \longrightarrow \mathcal{X}(\Gamma),\quad \mathsf{H}\colon \mathcal{V}_1 \longrightarrow \mathcal{C}(V) \;\text{ and }\; \mathsf{J}\colon \mathcal{V}_2 \longrightarrow \mathcal{C}(V).$$

We will say that F, G, H and J are *operators of order* n if n is the smallest non negative integer such that, for each $\mathsf{f} \in \mathcal{V}_1$ and each $u \in \mathcal{V}_2$ it is verified that

$$\begin{aligned}\mathsf{supp}(\mathsf{F}(\mathsf{f})),\ \mathsf{supp}(\mathsf{H}(\mathsf{f})) &\subset \{x \in V : d(x, \mathsf{supp}(\mathsf{f})) \le n\},\\ \mathsf{supp}(\mathsf{G}\,(u)),\ \mathsf{supp}(\mathsf{J}(u)) &\subset \{x \in V : d(x, \mathsf{supp}(u)) \le n\}.\end{aligned}$$

It is clear that the endomorphism of $\mathcal{X}(\Gamma)$ determined by a field of endomorphisms is a local operator, whereas the endomorphisms of $\mathcal{X}(\Gamma)$ that assign to each field its symmetric or its antisymmetric parts are both first order operators.

Since the function spaces $\mathcal{C}(V)$ and $\mathcal{C}(E)$ have finite dimension, each endomorphism of any of both spaces can be identified with a matrix of order $|V|$ or $|E|$ whose coefficients run over the vertices or edges. With these identifications in mind, S.P. Novikov introduced in [19] the notion of order of an endomorphism of the vertex and edge function spaces. His definition is different from ours since the former is determined by the maximal number of vertices (edges) in the shortest paths joining such pairs of vertices (edges) for which coefficients are nonzero. So, in Novikov's terminology the derivative operator, that we will introduced in the following section, has order two, whereas for us it has order one. Moreover, when the standard Laplacian of a graph is considered, Novikov's definition leads to a second order operator, but for us it is a first order operator. However, the Laplace operator of a network associated with a non orthogonal metric is a second order operator that would have third order with Novikov's definition.

It is clear that the composition of two homomorphisms of orders n and m respectively, is an homomorphism of order less or equal than $n+m$. Moreover, the notion of order is hereditary, in the sense that the restriction of an operator to a vector subspace has order less or equal than the one of the operator. This does not exclude that, in some cases, the order of the restriction could be strictly smaller.

3 Difference calculus on weighted networks

Our objective in this section is to consider general metrics on a graph which leads us to the concept of *(general) weighted network*. This structure together with suitable inner products on the space of vertex functions and on the space of vector fields allow to introduce the basic difference operators by means of duality techniques. We obtain such calculus starting from the derivative operator. This operator produces symmetric or antisymmetric vector fields depending on whether or not the manifold have been endowed with an orientation.

One of the most relevant difference operator is the combinatorial Laplacian which can be interpreted as the discrete counterpart of the Laplace-Beltrami operator of a compact Riemannian manifold. Although the combinatorial Laplacian has been widely used in relation with simple random walks, are the tools introduced by J. Dodziuk which allow to interpret it truly as a discrete version of the Laplace-Beltrami operator, see [11]. In this work standard inner products on the spaces of vertex functions and edge functions are considered and the combinatorial Laplacian is formally obtained as the composition of two difference operators which play the same role that the gradient and the divergence in the continuous case.

A function $\nu \in \mathcal{C}(V)$ is called a *weight on* V if $\nu(x) > 0$ for all $x \in V$. In particular we will denote by 1 the weight whose values equals 1 at any vertex. For each weight ν on V the expression $\displaystyle\int_V u\, v\, \nu\, dx = \sum_{x\in V} u(x)\, v(x)\, \nu(x)$, $u, v \in \mathcal{C}(V)$, determines an inner product on $\mathcal{C}(V)$. The corresponding Hilbert space will be denoted by $\mathsf{L}(\nu)$ and we drop ν when $\nu = 1$.

We call *weighted network,* a triple $(\Gamma, \mathsf{B}, \mu)$ where Γ is a graph, B is a metric on it and μ is a weight. Throughout this work, the role played by the weight μ introduced in the above definition will be shown clearly. In the first instance, it is worth to mention that its role can be resembled the Jacobian determinant of a parametrization of a differential manifold, as it is reflected on the following definition.

If $(\Gamma, \mathsf{B}, \mu)$ is a weighted network, the expression $\displaystyle\frac{1}{2}\int_V \mathsf{B}(\mathsf{f}, \mathsf{g})\, \mu\, dx$ determines an inner product on $\mathcal{X}(\Gamma)$, where the factor $\frac{1}{2}$ is due to the fact that each edge is considered twice. The vector space $\mathcal{X}(\Gamma)$ endowed with this inner product will be denoted by $\mathbf{H}(\mathsf{B}, \mu)$ or simply by $\mathbf{H}$ when B is the canonical metric and $\mu = 1$.

Observe that when B is an orthogonal metric with symmetric component function, then $\mathbf{H}(\mathsf{B}, \mu) = \frac{1}{\mu}\mathcal{X}^s(\Gamma) \oplus \mathcal{X}^a(\Gamma)$ and $\mathbf{H}(\mathsf{B}, \mu) = \mathcal{X}^s(\Gamma) \oplus \frac{1}{\mu}\mathcal{X}^a(\Gamma)$ are orthogonal decompositions. More generally, if B is an orthogonal metric whose component function is m, then for any $\mathsf{f}, \mathsf{g} \in \mathcal{X}^s(\Gamma)$ we get that

$$\frac{1}{2}\int_V \mathsf{B}(\mathsf{f}, \mathsf{g})\, \mu\, dx = \sum_{e\in E} r(e)\, f(e)\, g(e),$$

where $r = (\mu m)^s$ and $f, g \in \mathcal{C}(E)$ are identify with the component functions of f and g. Definitely, the inner product induced on $\mathcal{X}^s(\Gamma)$ as subspace of $\mathbf{H}(\mathsf{B}, \mu)$, corresponds to the standard inner product on $\mathcal{C}(E)$, with respect to the positive weight r, usually called *resistance*. Conversely, if r is a weight on E, then the inner product on $\mathcal{C}(E)$ with respect to r coincides with the induced by the orthogonal metric B whose component function is r. Therefore, in this case, the concept of weighted network introduced here coincides with the one established in the literature as *finite network* or, in the context of electrical networks, as *purely resistive network*, (see for instance, [22].) So, the inner product on $\mathcal{X}(\Gamma)$ defined in this work includes those inner products on $\mathcal{C}(E)$ that have been widely used in the context of networks, (see for instance [8, 18, 20, 22].)

3.1 First order difference operators

Our objective in this section is to define the discrete analogues of the fundamental first order differential operators on Riemannian manifolds, specifically the derivative, gradient and divergence operators. They will be generically called *first order difference operators on a weighted network.*

If Γ is a graph and τ is an orientation on it, we call derivative and oriented derivative of $u \in \mathcal{C}(V)$, the vector fields $\mathsf{d}u$ and $\mathsf{d}_\tau u$ given by

$$(\mathsf{d}u)(x) = \sum_{y \sim x} \Big(u(y) - u(x) \Big)\, e_{xy}, \quad (\mathsf{d}_\tau u)(x) = \sum_{y \sim x} \Big(u(\tau(e_{xy})) - u(\zeta(e_{xy})) \Big)\, e_{xy}, \tag{1}$$

respectively. In addition, the linear applications $\mathsf{d}, \mathsf{d}_\tau \colon \mathcal{C}(V) \longrightarrow \mathcal{X}(\Gamma)$ will be called *derivative* and *oriented derivative* operators. Moreover it is clear that $\mathsf{d}u = -2\, u^{\mathsf{a}}$, $\mathsf{d}_\tau u = \mathsf{D}_\tau \mathsf{d}u$ for any $u \in \mathcal{C}(V)$ and also that $\mathsf{Img}\, \mathsf{d} \subset \mathcal{X}^{\mathsf{a}}(\Gamma)$ and $\mathsf{Img}\, \mathsf{d}_\tau \subset \mathcal{X}^s(\Gamma)$.

Suppose now that the graph Γ is endowed with a metric B whose associated field of linear applications is M. We will call the operators $\nabla^{\mathsf{B}}, \nabla^{\mathsf{B}}_\tau \colon \mathcal{C}(V) \longrightarrow \mathcal{X}(\Gamma)$, given by $\mathsf{M}^{-1} \circ \mathsf{d}$ and $\mathsf{M}^{-1} \circ \mathsf{d}_\tau$ *gradient operator and oriented gradient operator with respect to* B, respectively. For each $u \in \mathcal{C}(V)$, the fields $\nabla^{\mathsf{B}} u$ and $\nabla^{\mathsf{B}}_\tau u$ will be called *gradient and oriented gradient of* u*, with respect to* B, respectively. Moreover a vector field f will be named *gradient field or oriented gradient field with respect to* B, if there exist $u \in \mathcal{C}(V)$ such that $\mathsf{f} = \nabla^{\mathsf{B}} u$ or $\mathsf{f} = \nabla^{\mathsf{B}}_\tau u$, respectively.

Next we describe some properties verified by the operators ∇^{B} and ∇^{B}_τ. Of course, all of them are also verified by the derivative and oriented derivative operators.

Proposition 3.1 *The following properties hold:*

i) The gradient and the oriented gradient operators are linear and of first order.

ii) If $u \in \mathcal{C}(V)$*, then* $\nabla^{\mathsf{B}} u = 0$ *or* $\nabla^{\mathsf{B}}_\tau u = 0$ *iff* u *is constant.*

iii) Leibnitz's Rule: *If* $u, v \in \mathcal{C}(V)$*, then*

$$\nabla^{\mathsf{B}}(uv) = v\, \nabla^{\mathsf{B}} u + u\, \nabla^{\mathsf{B}} v + \mathsf{M}^{-1}\mathsf{w} \quad \text{and} \quad \nabla^{\mathsf{B}}_\tau(uv) = v\, \nabla^{\mathsf{B}}_\tau u + u\, \nabla^{\mathsf{B}}_\tau v + \mathsf{M}^{-1}\mathsf{D}_\tau \mathsf{w},$$

where w *is the vector field whose component function is* $4u^{\mathsf{a}}v^{\mathsf{a}}$*. Moreover, if* B *is orthogonal, then*

$$\nabla^{\mathsf{B}}(uv) = v^s \cdot \nabla^{\mathsf{B}} u + u^s \cdot \nabla^{\mathsf{B}} v \quad \text{and} \quad \nabla^{\mathsf{B}}_\tau(uv) = v^s \cdot \nabla^{\mathsf{B}}_\tau u + u^s \cdot \nabla^{\mathsf{B}}_\tau v.$$

Proof. As the oriented case is easily deduced from the unoriented one, we only prove the unoriented version.

The linearity of the operators follows immediately. Moreover, ∇^{B} is a first order operator since $\mathsf{supp}(\mathsf{d}u) = \mathsf{supp}(u^{\mathsf{a}}) \subset \{x : d(x, \mathsf{supp}(u)) \leq 1\}$ and M^{-1} is a local operator. On the other hand, (ii) follows from a standard reasoning of connection. Finally, (iii) can be easily deduced from the identity $(uv)^{a} = u^{a}v^{s} + u^{s}v^{a}$. ∎

The Leibnitz's Rule here obtained is a generalization of the one obtained in [20], and independently in [21], for graphs.

Suppose now that besides the metric B we also consider on Γ two weights μ and ν. Then, ∇^{B} and $\nabla^{\mathsf{B}}_{\tau}$ can be seen as linear operators between the inner product spaces $\mathsf{L}(\nu)$ and $\mathbf{H}(\mathsf{B}, \mu)$. We will call *divergence* and *oriented divergence* operators the dual operators of gradient and oriented gradient, respectively. Moreover they will be denoted by $\mathsf{div}_{\nu\mu}$ and $\mathsf{div}_{\tau\nu\mu}$ and we will drop the subscripts when $\nu = \mu = 1$. For each $\mathsf{f} \in \mathcal{X}(\Gamma)$, the functions $\mathsf{div}_{\nu\mu}\mathsf{f}$ and $\mathsf{div}_{\tau\nu\mu}\mathsf{f}$ will be called *divergence and oriented divergence of* f, respectively. In particular, a vector field f is called *solenoidal* or *τ-solenoidal* if $\mathsf{div}_{\nu\mu}\mathsf{f} = 0$ or $\mathsf{div}_{\tau\nu\mu}\mathsf{f} = 0$, respectively.

Note that our definitions of oriented gradient and oriented divergence coincide basically with the ones that appear in [18], when $\nu = \mu = 1$ and B is an orthogonal metric whose component function is symmetric. In particular, when the metric is the canonical one, these expressions are the same that the given in [1, 12, 20, 21].

The operators $\mathsf{div}_{\nu\mu}, \mathsf{div}_{\tau\nu\mu}\colon \mathbf{H}(\mathsf{B}, \mu) \longrightarrow \mathsf{L}(\nu)$ are characterized by the following identities

$$\frac{1}{2}\int_V \mathsf{B}(\nabla^{\mathsf{B}}u, \mathsf{f})\,\mu\,dx = -\int_V u\,\mathsf{div}_{\nu\mu}\mathsf{f}\,\nu\,dx \quad\text{and}\quad \frac{1}{2}\int_V \mathsf{B}(\nabla^{\mathsf{B}}_{\tau}u, \mathsf{f})\,\mu\,dx = -\int_V u\,\mathsf{div}_{\tau\nu\mu}\mathsf{f}\,\nu\,dx, \tag{2}$$

that are valid for each $u \in \mathcal{C}(V)$ and each $\mathsf{f} \in \mathcal{X}(\Gamma)$.

Observe that as $\mathsf{B}(\nabla^{\mathsf{B}}u, \mathsf{f}) = \langle \mathsf{d}u, \mathsf{f}\rangle$, the operator $\mathsf{div}_{\nu\mu}$ does not depend on the metric B. In addition, as $\ker \mathsf{div}_{\nu\mu} = [\mathrm{Img}\,\nabla^{\mathsf{B}}]^{\perp}$ by virtue of Fredholm's Alternative, the space of solenoidal vector fields do not depend on the weight ν and the same occurs for the space of τ-solenoidal vector fields.

If we consider $u = 1$ in identities (2), then we obtain that

$$\int_V \mathsf{div}_{\nu\mu}\mathsf{f}\,\nu\,dx = \int_V \mathsf{div}_{\tau\nu\mu}\mathsf{f}\,\nu\,dx = 0. \tag{3}$$

In fact, this property characterize those functions that are the divergence or the oriented divergence of a vector field. Moreover the application of Fredholm's Alternative allows to obtain the following results.

Proposition 3.2 *Let $(\Gamma, \mathsf{B}, \mu)$ a weighted network and ν a weight on V. If $h \in \mathcal{C}(V)$, then there exists $\mathsf{f} \in \mathcal{X}(\Gamma)$ such that $\mathsf{div}_{\nu\mu}\mathsf{f} = h$, respectively such that $\mathsf{div}_{\tau\nu\mu}\mathsf{f} = h$, iff*

$\displaystyle\int_V h\,\nu\,dx = 0$. *In addition, it is verified that* $\dim \mathrm{Img}\,\mathsf{div}_{\nu\mu} = \dim \mathrm{Img}\,\mathsf{div}_{\tau\nu\mu} = |V|-1$ *and* $\dim\ker\mathsf{div}_{\nu\mu} = \dim\ker\mathsf{div}_{\tau\nu\mu} = |E|+1-\chi(\Gamma) \geq |E|$. *In particular,* Γ *is a tree iff* $\dim\ker\mathsf{div}_{\nu\mu} = |E|$ *or equivalently iff* $\dim\ker\mathsf{div}_{\tau\nu\mu} = |E|$.

Proposition 3.3 *Let* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *and* f *its component function, then for each* $x \in V$,

$$\mathsf{div}_{\nu\mu}\mathsf{f}(x) = \frac{1}{\nu(x)} \sum_{y\sim x} (\mu f)^{\mathrm{a}}(x,y)$$

and

$$\mathsf{div}_{\tau\nu\mu}\mathsf{f}(x) = \frac{1}{\nu(x)} \left[\sum_{\substack{y\sim x\\ \zeta(e_{xy})=x}} (\mu f)^{s}(x,y) - \sum_{\substack{y\sim x\\ \tau(e_{xy})=x}} (\mu f)^{s}(x,y) \right].$$

Therefore, $\mathsf{div}_{\nu\mu}\mathsf{f} = \frac{1}{\nu}\,\mathsf{div}(\mu\mathsf{f})$, $\mathsf{div}_{\tau\nu\mu}\mathsf{f} = \mathsf{div}_{\nu\mu}\mathsf{D}_\tau\mathsf{f}$ *for any* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *and* $\mathsf{div}_{\nu\mu}$ *and* $\mathsf{div}_{\tau\nu\mu}$ *are first order operators.*

Proof. We only prove the unoriented case since the proof for the oriented case follows the same arguments.

If $x \in V$ and ε_x denotes the Dirac measure at x, then applying (2) we obtain

$$\nu(x)\mathsf{div}_{\nu\mu}\mathsf{f}(x) = \int_V \varepsilon_x\, \mathsf{div}_{\nu\mu}\mathsf{f}\,\nu\,dy = -\frac{1}{2}\int_V \langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle\,\mu\,dy.$$

On the other hand, as $\mathsf{d}\varepsilon_x(x) = -\sum_{y\sim x} e_{xy}$, whereas $\mathsf{d}\varepsilon_x(y) = e_{xy}$ for each $y \sim x$, then $\langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle(x) = -\displaystyle\int_V f(x,y)\,dy$ and $\langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle(y) = f(y,x)$ for $y \neq x$. Therefore,

$$-\frac{1}{2}\int_V \langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle\,\mu\,dy = \frac{1}{2}\int_V \Big(\mu(x)f(x,y) - \mu(y)f(y,x)\Big)\,dy = \int_V (\mu f)^{\mathrm{a}}(x,y)\,dy.$$

Finally, it is clear that $\mathsf{supp}(\mathsf{div}_{\nu\mu}(\mathsf{f})) \subset \mathsf{supp}((\mu\mathsf{f})^{\mathrm{a}}) \subset \{x : d(x,\mathsf{supp}(\mathsf{f})) \leq 1\}$, which implies that $\mathsf{div}_{\nu\mu}$ is a first order operator on $\mathcal{X}(\Gamma)$. ∎

From the above proposition, we obtain that $\frac{1}{\mu}\mathcal{X}^s(\Gamma) \subset \ker\mathsf{div}_{\nu\mu}$ and $\frac{1}{\mu}\mathcal{X}^{\mathrm{a}}(\Gamma) \subset \ker\mathsf{div}_{\tau\nu\mu}$. In addition, Proposition 3.2 implies that any of these two inclusions is an equality iff Γ is a tree. In this case the Fredholm's Alternative allows us to obtain that f is a gradient field iff $\mathsf{Mf} \in \mathcal{X}^{\mathrm{a}}(\Gamma)$ or and oriented gradient field iff $\mathsf{Mf} \in \mathcal{X}^s(\Gamma)$.

A tree is the unique connected graph whose geometric realization is a simply connected unidimensional CW-complex, in fact it is contractible. So, the last result should be interpreted as the discrete analogue of the Poincaré Lemma. This motivates the following definition of curl operator on a weighted network.

If $\mathsf{f} \in \mathcal{X}(\Gamma)$, we call *curl and oriented curl of* f the vector fields given respectively by

$$\mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{f} = \frac{1}{\mu}\,(\mathsf{Mf})^s \quad \text{and} \quad \mathsf{curl}^{\mathsf{B}}_{\tau\mu}\mathsf{f} = \frac{1}{\mu}\,(\mathsf{Mf})^{\mathrm{a}}.$$

Moreover the associated endomorphisms $\mathsf{curl}^{\mathsf{B}}_{\mu}, \mathsf{curl}^{\mathsf{B}}_{\tau\mu}\colon \mathbf{H}(\mathsf{B},\mu) \longrightarrow \mathbf{H}(\mathsf{B},\mu)$ will be called the *curl* and the *oriented curl operators.* A vector field f is called *irrotational* or *oriented irrotational* if $\mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{f} = 0$ or $\mathsf{curl}^{\mathsf{B}}_{\tau\mu}\mathsf{f} = 0$, respectively.

Clearly both, curl and oriented curl, are first order operators and the oriented curl does not depend on the chosen orientation. Moreover, a vector field f is irrotational iff $\mathsf{Mf} \in \mathcal{X}^a(\Gamma)$ and it is oriented irrotational iff $\mathsf{Mf} \in \mathcal{X}^s(\Gamma)$. Therefore, $\mathsf{M}^{-1}(\mathcal{X}^a(\Gamma))$ and $\mathsf{M}^{-1}(\mathcal{X}^s(\Gamma))$ are the spaces of irrotational and oriented irrotational vector fields, respectively and therefore their dimensions equal $|E|$. In addition, it is verified that $\mathsf{M} = \mu\Big(\mathsf{curl}^{\mathsf{B}}_{\mu} + \mathsf{curl}^{\mathsf{B}}_{\tau\mu}\Big)$.

The following result shows that the definitions of the first order operators given in this section lead to a coherent vector calculus on weighted networks that is mimetic to its continuous counterpart.

Proposition 3.4 *The curl and the oriented curl are self-adjoint operators. Moreover, the following identities hold:*

$$\mathsf{div}_{\nu\mu} \circ \mathsf{curl}^{\mathsf{B}}_{\mu} = \mathsf{div}_{\tau\nu\mu} \circ \mathsf{curl}^{\mathsf{B}}_{\tau\mu} = 0 \quad \textit{and} \quad \mathsf{curl}^{\mathsf{B}}_{\mu} \circ \nabla^{\mathsf{B}} = \mathsf{curl}^{\mathsf{B}}_{\tau\mu} \circ \nabla^{\mathsf{B}}_{\tau} = 0.$$

Proof. As the oriented case is analogous to the unoriented one, we only prove the unoriented version. Let $\mathsf{f},\mathsf{g} \in \mathcal{X}(\Gamma)$, then

$$\begin{aligned}\int_V \mathsf{B}(\mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{f},\mathsf{g})\mu\,dx = & \int_V \langle \mathsf{M}\mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{f},\mathsf{g}\rangle \mu\,dx = \int_V \langle (\mathsf{Mf})^s, \mathsf{Mg}\rangle\,dx = \int_V \langle (\mathsf{Mf})^s, (\mathsf{Mg})^s\rangle\,dx \\ = & \int_V \langle \mathsf{Mf}, \mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{g}\rangle \mu\,dx = \int_V \mathsf{B}(\mathsf{f},\mathsf{curl}^{\mathsf{B}}_{\mu}\mathsf{g})\mu\,dx.\end{aligned}$$

The rest of the claims follows directly from the definition of the involved operators. ∎

Corollary 3.5 (Poincaré Lemma.) *The underlaying graph to a weighted network is a tree iff each irrotational field is a gradient field or, equivalently, iff each solenoidal field is the curl of another field. An analogous result is true in the oriented case.*

Proof. From the above proposition, the space of irrotational fields coincides with the space of gradient fields iff $|E| = \dim \mathrm{Img} \nabla^{\mathsf{B}} = \dim\, [\ker \mathsf{div}_{\nu\mu}]^{\perp}$, that is iff $\dim \ker \mathsf{div}_{\nu\mu} = |E|$, whereas the space of solenoidal fields equals the space of the vector fields that are the curl of another vector field iff $|E| = \dim \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}} = \dim \ker \mathsf{div}_{\nu\mu}$. In both cases the conclusion follows from Proposition 3.2. ∎

We finish this paragraph by introducing another first order operator that is an endomorphism of $\mathsf{L}(\nu)$ and that will be important in applications. Specifically, fixed a vector field f we define the *directional derivative along* f of a vertex function u by $\dfrac{\partial u}{\partial \mathsf{f}}(x) = \langle \mathsf{f}, \mathsf{d}u\rangle(x)$

for any $x \in V$. Clearly, the *derivative along* f, $\dfrac{\partial}{\partial \mathsf{f}}$, is a first order operator viewed as an endomorphism of $\mathsf{L}(\nu)$. The following result can be considered as a first discrete version of the Integration by Parts technique.

Proposition 3.6 *Given* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *whose component function is* $f \in \mathcal{C}(\Gamma)$*, then for any functions* $u, v \in \mathcal{C}(V)$*, it is verified that*

$$\int_V v \frac{\partial u}{\partial \mathsf{f}}\, \nu\, dx = -\frac{1}{2}\int_{V\times V} (\nu f)^s(x,y)\, \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, dxdy - \int_V (\mathsf{div}_{\nu\nu}\mathsf{f})\, uv\, \nu\, dx$$
$$- \int_{V\times V} (\nu f)^{\mathsf{a}}(x,y)\, u(x)\, v(y)\, dxdy.$$

Proof. Firstly, we get that $\dfrac{\partial u}{\partial \mathsf{f}}(x) = \displaystyle\int_V f(x,y)\big(u(y)-u(x)\big)\, dy$, for any $x \in V$. Therefore,

$$\begin{aligned}\int_V v \frac{\partial u}{\partial \mathsf{f}}\, \nu\, dx &= \int_{V\times V} \nu(x) f(x,y) v(x)\big(u(y)-u(x)\big)\, dydx \\ &= \frac{1}{2}\int_{V\times V} \nu(x) f(x,y)\big(u(y)-u(x)\big)\big(v(x)-v(y)\big)\, dydx \\ &+ \int_{V\times V} (\nu f)^{\mathsf{a}}(x,y)\, u(y)\, v(y)\, dxdy - \int_{V\times V} (\nu f)^{\mathsf{a}}(x,y)\, u(x)\, v(y)\, dxdy.\end{aligned}$$

Moreover, $\displaystyle\int_{V\times V} (\nu f)^{\mathsf{a}}(x,y)\, u(y)\, v(y)\, dxdy = -\int_V (\mathsf{div}_{\nu\nu}\mathsf{f})\, uv\, \nu\, dy$ and the result follows tacking into account that

$$\begin{aligned}&\int_{V\times V} \nu(x) f(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)dydx \\ &= \int_{V\times V} (\nu f)^s(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)dydx. \quad \blacksquare\end{aligned}$$

As a consequence of the above proposition we determine the adjoint operator of the derivative along f when it is considered as an endomorphism of $\mathsf{L}(\nu)$.

Corollary 3.7 *Given* $\mathsf{f} \in \mathcal{X}(\Gamma)$*, then*

$$\left(\frac{\partial}{\partial \mathsf{f}}\right)^* = \frac{\partial}{\partial \hat{\mathsf{f}}} - 2\,\mathsf{div}_{\nu\nu}\mathsf{f}$$

where $\hat{\mathsf{f}} = \dfrac{1}{\nu}\Big((\nu\,\mathsf{f})^s - (\nu\,\mathsf{f})^{\mathsf{a}}\Big)$*. In particular, the directional derivative along* f *is a self-adjoint operator on* $\mathsf{L}(\nu)$ *iff* $\nu\,\mathsf{f} \in \mathcal{X}^s(\Gamma)$*.*

Proof. Keeping in mind that $(\nu\,\hat{\mathsf{f}})^s = (\nu\,\mathsf{f})^s$ and $(\nu\,\hat{\mathsf{f}})^a = -(\nu\,\mathsf{f})^a$ and applying the above proposition to the field $\hat{\mathsf{f}}$, for any $u, v \in \mathsf{L}(\nu)$ we obtain that

$$\int_V u\frac{\partial v}{\partial\hat{\mathsf{f}}}\,\nu\,dx = -\frac{1}{2}\int_{V\times V}(\nu f)^s(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dxdy + \int_V(\mathsf{div}_{\nu\nu}\mathsf{f})\,uv\,\nu\,dx$$
$$+\int_{V\times V}(\nu f)^a(x,y)\,v(x)\,u(y)\,dxdy.$$

On the other hand, as $\displaystyle\int_{V\times V}(\nu f)^a(x,y)\,v(x)\,u(y)\,dxdy = -\int_{V\times V}(\nu f)^a(x,y)\,u(x)\,v(y)\,dxdy$, the above identity shows that

$$\int_V u\frac{\partial v}{\partial\hat{\mathsf{f}}}\,\nu\,dx = \int_V v\frac{\partial u}{\partial\mathsf{f}}\,\nu\,dx + 2\int_V(\mathsf{div}_{\nu\nu}\mathsf{f})\,uv\,\nu\,dx,$$

that proves the first claim.

Finally, as $\hat{\mathsf{f}} = \mathsf{f} - \dfrac{2}{\nu}\,(\nu\,\mathsf{f})^a$, then $\dfrac{\partial u}{\partial\hat{\mathsf{f}}} = \dfrac{\partial u}{\partial\mathsf{f}} - \dfrac{2}{\nu}\,\dfrac{\partial u}{\partial(\nu\,\mathsf{f})^a}$ and therefore $\dfrac{\partial}{\partial\mathsf{f}}$ is self-adjoint iff $\dfrac{\partial u}{\partial(\nu\,\mathsf{f})^a} = -u\,\nu\,\mathsf{div}_{\nu\nu}\mathsf{f}$ for any $u \in \mathcal{C}(V)$. Clearly the above identity is verified when $\nu\,\mathsf{f} \in \mathcal{X}^s(\Gamma)$. Conversely, if the equality is true for any $u \in \mathcal{C}(V)$, tacking any non null constant we obtain that necessarily $\mathsf{div}_{\nu\nu}\mathsf{f} = 0$ and hence that $\dfrac{\partial u}{\partial(\nu\,\mathsf{f})^a} = 0$ for any $u \in \mathcal{C}(V)$. If we consider now $u = \varepsilon_y$, $y \in V$, the above identity says that $\dfrac{\partial u}{\partial(\nu\,\mathsf{f})^a}(x) = (\nu f)^a(x,y) = 0$ for all $x \in V\setminus\{y\}$. In conclusion, $(\nu\,\mathsf{f})^a = \mathsf{0}$, that is, $\nu\,\mathsf{f} \in \mathcal{X}^s(\Gamma)$. ∎

If we consider d_τ instead of d in the definition of the derivative along the vector field f, then $\dfrac{\partial}{\partial_\tau\mathsf{f}} = \dfrac{\partial}{\partial\mathsf{D}_\tau\mathsf{f}}$. Therefore, $\left(\dfrac{\partial}{\partial_\tau\mathsf{f}}\right)^* = -\dfrac{\partial}{\partial_\tau\hat{\mathsf{f}}} - 2\,\mathsf{div}_{\tau\nu\nu}\mathsf{f}$ and hence $\dfrac{\partial}{\partial_\tau\mathsf{f}}$ is self-adjoint iff $\nu\,\mathsf{f} \in \mathcal{X}^a(\Gamma)$.

3.2 Second order difference operators

In this section we introduce the fundamental second order difference operators on the vertex function space of a weighted network which are obtained by composition of two first order operators. From now on we suppose fixed the canonical metric, ν and μ weights on V and τ an orientation on Γ. For each field of endomorphisms A consider the endomorphisms of $\mathsf{L}(\nu)$ given by $\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u) = -\mathsf{div}_{\nu\mu}(\mathsf{A}\mathsf{d}u)$ and $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}(u) = -\mathsf{div}_{\tau\nu\mu}(\mathsf{A}\mathsf{d}_\tau u)$ for all $u \in \mathsf{L}(\nu)$. Clearly, both operators have second order at most and they are related by the identity $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu} = \mathcal{L}^{\mathsf{A}_\tau}_{\nu\mu}$, where $\mathsf{A}_\tau = \mathsf{D}_\tau\mathsf{A}\mathsf{D}_\tau$. In particular, when A is a diagonal field of endomorphisms, then $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu} = \mathcal{L}^{\mathsf{A}}_{\nu\mu}$.

Proposition 3.8 *For each $u, v \in \mathcal{C}(V)$, the following* Green's Identities *are verified*

$$\int_V v\,\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx = \frac{1}{2}\int_V\langle\mathsf{A}\mathsf{d}u,\mathsf{d}v\rangle\,\mu\,dx \quad \text{and} \quad \int_V v\,\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}(u)\,\nu\,dx = \frac{1}{2}\int_V\langle\mathsf{A}\mathsf{d}_\tau u,\mathsf{d}_\tau v\rangle\,\mu\,dx.$$

In particular, $\displaystyle\int_V \mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx = \int_V \mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}(u)\,\nu\,dx = 0$ *for each* $u \in \mathcal{C}(V)$ *and* $\mathcal{L}^{\mathsf{A}^t}_{\nu\mu}$ *and* $\mathcal{L}^{\mathsf{A}^t}_{\tau\nu\mu}$ *are the adjoint operators of* $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ *and* $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}$ *on* $\mathsf{L}(\nu)$*, respectively. Moreover, the operators* $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ *and* $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}$ *have first order when* A *is a diagonal field of endomorphisms.*

Proof. Newly, it suffices to prove the unoriented case. The Green's Identity and property $\displaystyle\int_V \mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx = 0$ are straightforward consequences of the definition of the divergence operator and identity (3). In particular, by applying the Green's identity to $\mathcal{L}^{\mathsf{A}^t}_{\nu\mu}$, we obtain that for each $u, v \in \mathsf{L}(\nu)$

$$\int_V u\,\mathcal{L}^{\mathsf{A}^t}_{\nu\mu}(v)\,\nu = \frac{1}{2}\int_V \langle \mathsf{A}^t\mathsf{d}v, \mathsf{d}u\rangle\,\mu\,dx = \frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}u, \mathsf{d}v\rangle\,\mu\,dx = \int_V v\,\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx.$$

Finally, if A is a diagonal field of endomorphisms, then $\mathsf{supp}(\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)) \subset \mathsf{supp}((\mu u)^a)$, since in this case $(\mu\mathsf{A}\mathsf{d}u)^a = -2\,a^s\,(\mu u)^a$, where a is the component function of A. This implies that $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is a first order operator. ∎

The above identities lead us to generalize the concept of elliptic operator introduced by Y. Colin de Verdière in [10]. So, we will say that the operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is *semielliptic* when it is selft-adjoint and positive semidefinite and *elliptic* when, in addition $\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u) = 0$ iff u is a constant function. Clearly, the above notions do not depend on the weight ν and for this reason the pair (A, μ) will be called semi-elliptic or elliptic when the operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is semi-elliptic or elliptic, respectively. The same definitions apply for the oriented version.

We remark that when $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is self-adjoint then $\mathcal{L}^{\mathsf{A}}_{\nu\mu} = \mathcal{L}^{\mathsf{A}^t}_{\nu\mu}$ and hence we can suppose without loss of generality that the field of endomorphisms A is symmetric. Moreover if A is a symmetric and positive semi-definite field of endomorphisms, then the pair (A, μ) is semi-elliptic for any weight μ and it is elliptic if in addition A is a positive definite field. Of course, when A is a symmetric and positive definite field of endomorphisms, the field A^{-1} has the same properties and both, A and A^{-1}, can be considered as metrics on Γ. So, we can associate with each metric and with each orientation on Γ an elliptic operator. Specifically, let $(\Gamma, \mathsf{B}, \mu)$ be a weighted network, τ an orientation on Γ and ν a weight on V. We will call *Laplace operator and oriented Laplace operator*, the linear operators $\Delta^{\mathsf{B}}_{\nu\mu}, \Delta^{\mathsf{B}}_{\tau\nu\mu}\colon \mathsf{L}(\nu) \longrightarrow \mathsf{L}(\nu)$ given by $\Delta^{\mathsf{B}}_{\nu\mu} = \mathsf{div}_{\nu\mu} \circ \nabla^{\mathsf{B}}$ and $\Delta^{\mathsf{B}}_{\tau\nu\mu} = \mathsf{div}_{\tau\nu\mu} \circ \nabla^{\mathsf{B}}_{\tau}$, respectively. In addition, for each $u \in \mathsf{L}(\nu)$, the functions $\Delta^{\mathsf{B}}_{\nu\mu}u$ and $\Delta^{\mathsf{B}}_{\tau\nu\mu}u$ will be called *Laplacian and oriented Laplacian of* u, respectively.

The following properties are a direct consequence of Proposition 3.8.

Proposition 3.9 *If* $(\Gamma, \mathsf{B}, \mu)$ *is a weighted network,* τ *an orientation on* Γ *an* ν *a weight on* V*, then the following properties hold:*

i) The Laplace and the oriented Laplace operators have order 2 *at most. Moreover, they are first order operators when* B *is an orthogonal metric and in this case both operator coincide.*

ii) (Green's Identities): *For each* $u, v \in \mathcal{C}(V)$, $\displaystyle\int_V v\, \Delta^{\mathsf{B}}_{\nu\mu} u\, \nu\, dx = -\frac{1}{2}\int_V \mathsf{B}(\nabla^{\mathsf{B}} u, \nabla^{\mathsf{B}} v)\, \mu\, dxdy$ *and* $\displaystyle\int_V v\, \Delta^{\mathsf{B}}_{\tau\nu\mu} u\, \nu\, dx = -\frac{1}{2}\int_V \mathsf{B}(\nabla^{\mathsf{B}}_\tau u, \nabla^{\mathsf{B}}_\tau v)\, \mu\, dxdy$. *In particular, the Laplace and the oriented Laplace operators are self-adjoint on* $\mathsf{L}(\nu)$.

iii) (Gauss's Theorem): *For each* $u \in \mathcal{C}(V)$, $\displaystyle\int_V \Delta^{\mathsf{B}}_{\nu\mu} u\, \nu\, dx = \int_V \Delta^{\mathsf{B}}_{\tau\nu\mu} u\, \nu\, dx = 0.$

iv) The Laplacian and the oriented Laplacian are elliptic operators.

Our next aim is to obtain an explicit expression of $\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)$. Of course, we can deduce the corresponding expression for $\mathcal{L}^{\mathsf{A}}_{\tau\nu\mu}(u)$ by changing A by $\mathsf{D}_\tau \mathsf{A} \mathsf{D}_\tau$ and taking into account that if the component function of A is $a \in \mathcal{C}(\Gamma \times \Gamma)$, then the component function of $\mathsf{D}_\tau \mathsf{A} \mathsf{D}_\tau$ is given by $a_\tau(x, y, z) = a(x, y, z)$ when $x = \tau(e_{xy}) = \tau(e_{xz})$ or $x = \zeta(e_{xy}) = \zeta(e_{xz})$ and by $a_\tau(x, y, z) = -a(x, y, z)$, otherwise.

From Proposition 3.8, and keeping in mind that $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is a linear operator, it is clear that

$$\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)(x) = \frac{1}{2\nu(x)} \sum_{y \in V} u(y) \int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x \rangle\, \mu\, dz, \text{ for any } x \in V.$$

Therefore, we define the *coefficient function of the pair* (A, μ) as the function $c^{\mathsf{A}}_\mu \colon V \times V \longrightarrow \mathbb{R}$ given by

$$c^{\mathsf{A}}_\mu(x, y) = -\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x \rangle\, \mu\, dz, \quad \text{if } x \neq y \text{ and } c^{\mathsf{A}}_\mu(x, x) = 0, \quad x \in V. \tag{4}$$

Moreover, $\displaystyle\int_V c^{\mathsf{A}}_\mu(x, y)\, dy = \frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_x, \mathsf{d}\varepsilon_x \rangle\, \mu\, dz$, since $c^{\mathsf{A}}_\mu(x, y) = -\nu(y)\mathcal{L}^{\mathsf{A}^t}_{\nu\mu}(\varepsilon_x)(y)$, when $x \neq y$. Hence,

$$\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)(x) = \frac{1}{\nu(x)}\int_V c^{\mathsf{A}}_{\nu\mu}(x, y) \big(u(x) - u(y)\big)\, dy, \qquad x \in V. \tag{5}$$

Lemma 3.10 *Let* $\mathsf{A} \in \mathcal{T}^1(\Gamma)$, $a \in \mathcal{C}(\Gamma \times \Gamma)$ *its component function and* μ *a weight on* V. *Then, the coefficient function of the pair* (A, μ) *is given by*

$$c^{\mathsf{A}}_\mu(x, y) = \frac{1}{2}\int_V \Big[\mu(x)a(x, z, y) + \mu(y)a(y, x, z) - \mu(z)a(z, x, y)\Big]\, dz, \qquad x \neq y.$$

In particular, $c^{\mathsf{A}^t}_\mu(x, y) = c^{\mathsf{A}}_\mu(y, x)$ *and hence* c^{A}_μ *is symmetric when* A *is symmetric. In addition, for any* $x \in V$ *we get that*

$$\int_V c^{\mathsf{A}}_\mu(x, y)\, dy = \int_V c^{\mathsf{A}}_\mu(y, x)\, dy = \frac{\mu(x)}{2}\langle \mathsf{A}1, 1\rangle(x) + \frac{1}{2}\int_V \mu(z)a(z, x, x)\, dz.$$

Proof. For any $x \in V$, the component function of the vector field $\mathsf{h}_x = \mathsf{d}\varepsilon_x$ is given by $h_x(x,z) = -1$, if $z \sim x$, $h_x(y,x) = 1$, if $y \sim x$ and $h_x(y,z) = 0$, otherwise. Moreover, if $\mathsf{f} \in \mathcal{X}(\Gamma)$ then $\langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle(x) = -\displaystyle\int_V f(x,y)\,dy$, whereas $\langle \mathsf{d}\varepsilon_x, \mathsf{f}\rangle(y) = f(y,x)$ if $y \neq x$. Therefore, the component function of the vector field $\mathsf{g}_y = \mathsf{A}\mathsf{d}\varepsilon_y$ is given by $g_y(w,t) = a(w,t,y)$, if $w \neq y$ and $g_y(y,t) = -\displaystyle\int_V a(y,t,z)\,dz$, which implies that

$$\langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle(z) = g_y(z,x), \quad \text{if} \quad z \neq x \quad \text{and} \quad \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle(x) = -\int_V g_y(x,t)\,dt$$

and hence

$$\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle\,\mu\,dz = -\frac{\mu(x)}{2}\int_V g_y(x,t)\,dt + \frac{1}{2}\int_V \mu(z)\,g_y(z,x)\,dz.$$

So, if $x \neq y$

$$\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_y, \mathsf{d}\varepsilon_x\rangle\,\mu\,dz = -\frac{\mu(x)}{2}\int_V a(x,t,y)\,dt - \frac{\mu(y)}{2}\int_V a(y,x,z)\,dz + \frac{1}{2}\int_V \mu(z)\,a(z,x,y)\,dz,$$

whereas

$$\begin{aligned}\frac{1}{2}\int_V \langle \mathsf{A}\mathsf{d}\varepsilon_x, \mathsf{d}\varepsilon_x\rangle\,\mu\,dz &= \frac{\mu(x)}{2}\int_V\int_V a(x,t,z)\,dzdt + \frac{1}{2}\int_V \mu(z)\,a(z,x,x)\,dz \\ &= \frac{\mu(x)}{2}\langle \mathsf{A}1, 1\rangle(x) + \frac{1}{2}\int_V \mu(z)\,a(z,x,x)\,dz.\end{aligned}$$

The identity $c_\mu^{\mathsf{A}}(x,y) = c_\mu^{\mathsf{A}^t}(y,x)$, $x,y \in V$, follows tacking into account that if a^t denotes the component function of A^t, then $a^t(x,y,z) = a(x,z,y)$ for any $x,y,z \in V$. ∎

Note that $c_\mu^{\mathsf{A}}(x,y) = 0$ when $d(x,y) > 2$, $c_\mu^{\mathsf{A}}(x,y) = -\dfrac{1}{2}\displaystyle\int_V \mu(z)a(z,x,y)\,dz$ if $d(x,y) = 2$, whereas $c_\mu^{\mathsf{A}}(x,y) = \dfrac{1}{2}\displaystyle\int_V \Big[\mu(x)a(x,z,y) + \mu(y)a(y,x,z)\Big]\,dz$ if $d(x,y) = 1$ and x and y do not belong to any triangle. Moreover, if $\mathsf{A} = \mathsf{D}_{\mathsf{f}}$ with $\mathsf{f} \in \mathcal{X}(\Gamma)$, then $c_\mu^{\mathsf{A}} = (\mu f)^s$, where $f \in \mathcal{C}(\Gamma)$ is the component function of f.

Given $\mathsf{A} \in \mathcal{T}^1(\Gamma)$, we can consider the symmetric and skew-symmetric parts of A, that are the fields of endomorphisms $\mathsf{A}_s, \mathsf{A}_{\mathsf{a}}$ defined respectively by $\mathsf{A}_s(x) = \frac{1}{2}(\mathsf{A}(x) + \mathsf{A}^t(x))$ and $\mathsf{A}_{\mathsf{a}}(x) = \frac{1}{2}(\mathsf{A}(x) - \mathsf{A}^t(x))$ for any $x \in V$. Moreover, the functions $c_\mu^{\mathsf{A}_s} = \frac{1}{2}(c_\mu^{\mathsf{A}} + c_\mu^{\mathsf{A}^t})$ and $c_\mu^{\mathsf{A}_{\mathsf{a}}} = \frac{1}{2}(c_\mu^{\mathsf{A}} - c_\mu^{\mathsf{A}^t})$ allow us to describe accurately the bilinear form on $\mathsf{L}(\nu)$ associated with the operator $\mathcal{L}_{\nu\mu}^{\mathsf{A}}$.

Proposition 3.11 *If $\mathsf{A} \in \mathcal{T}^1(\Gamma)$, then for all $u, v \in \mathsf{L}(\nu)$ the following identity holds*

$$\begin{aligned}\int_V v\,\mathcal{L}_{\nu\mu}^{\mathsf{A}}(u)\,\nu\,dx = \frac{1}{2}&\int_{V\times V} c_\mu^{\mathsf{A}_s}(x,y)\,\big(u(x) - u(y)\big)\,\big(v(x) - v(y)\big)\,dxdy \\ &+ \int_{V\times V} c_\mu^{\mathsf{A}_{\mathsf{a}}}(x,y)\,u(x)\,v(y)\,dxdy.\end{aligned}$$

Proof. From the expression of operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ given in (5), we get that

$$\int_V v\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx = \int_{V\times V} c^{\mathsf{A}}_\mu(x,y)\,v(x)\,\Big(u(x)-u(y)\Big)\,dydx$$

and hence

$$\begin{aligned}\int_V v\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx &= \frac{1}{2}\int_{V\times V} c^{\mathsf{A}}_\mu(x,y)\,v(x)\,\Big(u(x)-u(y)\Big)\,dydx\\ &-\frac{1}{2}\int_{V\times V} c^{\mathsf{A}}_\mu(y,x)\,v(y)\,\Big(u(x)-u(y)\Big)\,dxdy\\ &=\frac{1}{2}\int_{V\times V} c^{\mathsf{A}}_\mu(x,y)\,\Big(u(x)-u(y)\Big)\,\Big(v(x)-v(y)\Big)\,dydx\\ &+\frac{1}{2}\int_{V\times V}\Big(c^{\mathsf{A}}_\mu(x,y)-c^{\mathsf{A}}_\mu(y,x)\Big)\,v(y)\,\Big(u(x)-u(y)\Big)\,dxdy\\ &=\frac{1}{2}\int_{V\times V} c^{\mathsf{A}s}_\mu(x,y)\,\Big(u(x)-u(y)\Big)\,\Big(v(x)-v(y)\Big)\,dydx\\ &+\int_{V\times V} c^{\mathsf{A}\mathsf{a}}_\mu(x,y)\,u(x)\,v(y)\;dxdy,\end{aligned}$$

since from Lemma 3.10, $\displaystyle\int_V c^{\mathsf{A}\mathsf{a}}_\mu(x,y)\,dx = 0$ and where we have taken into account that

$$\int_{V\times V} c^{\mathsf{A}}_\mu(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)dxdy = \int_{V\times V} c^{\mathsf{A}s}_\mu(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)dxdy,$$

for any $u,v\in\mathcal{C}(V)$. ∎

The above proposition implies that the bilinear form $\displaystyle\int_V v\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx$ is symmetric iff $c^{\mathsf{A}}_\mu = c^{\mathsf{A}^t}_\mu$. In this case, $\mathcal{L}^{\mathsf{A}}_{\nu\mu} = \mathcal{L}^{\mathsf{A}s}_{\nu\mu}$ and hence we can suppose that A is a symmetric field of endomorphisms. So, the pair (A,μ) is semi-elliptic iff $c^{\mathsf{A}}_\mu(x,y) = c^{\mathsf{A}}_\mu(y,x)$ for all $x,y\in V$ and in addition

$$\int_{V\times V} c^{\mathsf{A}}_\mu(x,y)\,\Big(u(x)-u(y)\Big)^2 dxdy \geq 0, \qquad \text{for any}\;\; u\in\mathcal{C}(V).$$

We remark that above inequality, does not imply the non negativeness of the function c^{A}_μ. For instance, if we consider $K_3=\{x_1,x_2,x_3\}$ the complete graph, $\mu=1$ and $\mathsf{A}=\mathsf{D}_\mathsf{f}$ the diagonal field of endomorphisms whose component function is given by

$$f(x_1,x_2)=f(x_2,x_1)=3,\;\; f(x_2,x_3)=f(x_3,x_2)=2 \;\text{ and }\; f(x_1,x_3)=f(x_3,x_1)=-1,$$

then $c^{\mathsf{A}}_\mu(x,y)=f(x,y)$, for any $x,y\in K_3$ and the pair (A,μ) is elliptic.

On the other hand, it is clear that if $c^{\mathsf{A}}_\mu = c^{\mathsf{A}^t}_\mu$ and c^{A}_μ is a non negative function, then the pair (A,μ) is semi-elliptic. Moreover if, in addition, $c^{\mathsf{A}}_\mu(x,y)>0$ when $x\sim y$, then the pair (A,μ) is elliptic. We will say that the operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$, or equivalently the pair (A,μ), is *strongly elliptic*, if it is self-adjoint, $c^{\mathsf{A}}_{\mu}\geq 0$ and moreover $c^{\mathsf{A}}_{\mu}(x,y)>0$ when $x\sim y$.

Observe that when $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is a strongly elliptic operator, then $c_0=\frac{1}{2}\min\{c^{\mathsf{A}}_{\mu}(x,y):x\sim y\}$ satisfies that $c_0>0$ and hence we get that

$$\int_V u\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx\geq c_0\int_V\langle \mathsf{d}u,\mathsf{d}u\rangle\,dx,\qquad \text{for any}\ \ u\in\mathcal{C}(V).$$

In view of applications, it is of interest to describe when the pair (A,μ) is strongly elliptic. The following result establishes simple conditions on the component function of A to ensure this property, independently of the weight μ.

Lemma 3.12 *Let* A *a symmetric field of endomorphisms and* $a\in\mathcal{C}(\Gamma\times\Gamma)$ *its component function. If* $a(z,x,y)\leq 0$ *for all* $x,y,z\in V$ *with* $x\neq y$ *and* $\displaystyle\int_V a(x,y,z)\,dz\geq 0$ *for all* $x,y\in V$*, then* $c^{\mathsf{A}}_{\mu}\geq 0$ *for any weight* μ*. In addition, if when* $x\sim y$ *it is verified that either* $\displaystyle\int_V a(x,y,z)\,dz>0$ *or there exists* $z\in V$ *such that* $a(z,x,y)<0$*, then* $c^{\mathsf{A}}_{\mu}(x,y)>0$ *for any weight* μ*.*

Observe that if for each $x\in V$ we consider the symmetric matrix of order $k(x)$ given by $A(x)=\Big(a(x,y,z)\Big)_{\substack{y\sim x\\ z\sim x}}$, then the hypotheses of the above lemma say nothing else than $A(x)$ is a diagonally dominant M-matrix.

We conclude this section with some remarks. Firstly, when $\nu=\mu=1$, the definition of the Laplace operator of a weighted network is the discrete analogue of the Laplace operator of a differentiable Riemannian manifold, whereas the case $\nu=\mu$ corresponds to the expression of this operator in coordinates, where μ plays the role of the module of the Jacobian determinant. In general, the Laplace operator can also be interpreted as the analogue discrete of the Laplace-Beltrami operator of a *weighted Riemannian manifold*, see for instance [2]. In this ambit, the discrete operators studied in the literature basically correspond to the case in which the field A correspond to an orthogonal metric and $\mu=1$. The particular case $\nu=1$, leads to the so-called *combinatorial Laplacian*, whereas when $\nu(x)=\displaystyle\int_V c^{\mathsf{A}}_{\mu}(x,y)\,dy$, then the corresponding Laplace operator is the so-called *probabilistic Laplacian*, which is associated with a reversible random walk whose stationary distribution is ν, see for instance [8]. Of course, the above concept can be extended to general metrics as follows: if we suppose that the pair (A,μ) is strongly-elliptic, then we can define the probabilistic Laplacian by considering ν as before. In this case, the associated reversible random walk is not necessarily of nearest neighbor type.

In the electrical network context, if $\mathsf{A}\in\mathcal{T}^1(\Gamma)$ the expression $\mathsf{f}=\mathsf{A}\mathsf{d}u$ can be interpreted as a general linear *Ohm's Law* described in terms of the *admittance field* A, where u represent the *potential*, $\mathsf{d}u$ the *voltage* and f the *current* of the network. Therefore for any *current source* $g\in\mathcal{C}(V)$, the identity $\mathsf{div}_{\nu\mu}\mathsf{f}=g$, that is $\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)=-g$, represent the state equation of the network, obtained by application of the *Kirchhoff's Laws*, and then c^{A}_μ is nothing else than the *conductance function* of the network.

4 The cohomology of a weighted network

We aim here at obtaining some results about weighted networks that classically fall within the ambit of Algebraic Topology. The consideration in network analysis of the terminology and techniques from Algebraic Topology has provided a rigorous treatment of infinite networks and it has become standard (see for instance [19, 22].) However, this is not the case of finite network and the unique mention to this language is the definition of *cohomology of a multigraph* that can be found in [7], see also [1].

Our main goal is to show that the status of weighted networks is mimetic to the one of compact differentiable Riemannian manifolds: The discrete analogue of the *De Rham cohomology* gives the fundamental properties of the ordinary cohomology in the ambit of Algebraic Topology. On the other hand, this development will confirm the appropriateness of the definitions of the difference operators on a weighted network given in the preceding sections. These definitions allow to consider the Hodge Laplacian on the space of vector fields of the manifold. Although this situation is not new, (see for instance [11, 15, 21]), the Hodge Laplacian is obtained in the above-mentioned works by considering the network as the 1-skeleton of a simplicial complex which dimension equals to the one of the ambient space. In contrast, the results presented in this section do not need to consider the underlying graph as a part of a higher dimensional CW-complex.

Let $(\Gamma, \mathsf{B}, \mu)$ be a weighted network, ν a weight on V and τ a fixed orientation on Γ. We consider the (exterior) derivative operators $\mathsf{d}_{-1} = \mathsf{0}$, $\mathsf{d}_0 = \nabla^{\mathsf{B}}$ and $\mathsf{d}_1 = \mathsf{curl}^{\mathsf{B}}_\mu$ and its oriented versions $\mathsf{d}_{\tau_{-1}} = \mathsf{0}$, $\mathsf{d}_{\tau_0} = \nabla^{\mathsf{B}}_\tau$ and $\mathsf{d}_{\tau_1} = \mathsf{curl}^{\mathsf{B}}_{\tau\mu}$. Tacking into account that $\mathsf{d}_n \circ \mathsf{d}_{n-1} = \mathsf{d}_{\tau_n} \circ \mathsf{d}_{\tau_{n-1}} = \mathsf{0}$, $n = 0, 1$, we can define the complexes

$$\mathsf{0} \overset{\mathsf{d}_{-1}}{\hookrightarrow} \mathsf{L}(\nu) \xrightarrow{\mathsf{d}_0} \mathbf{H}(\mathsf{B}, \mu) \xrightarrow{\mathsf{d}_1} \mathbf{H}(\mathsf{B}, \mu) \quad \text{and} \quad \mathsf{0} \overset{\mathsf{d}_{\tau_{-1}}}{\hookrightarrow} \mathsf{L}(\nu) \xrightarrow{\mathsf{d}_{\tau_0}} \mathbf{H}(\mathsf{B}, \mu) \xrightarrow{\mathsf{d}_{\tau_1}} \mathbf{H}(\mathsf{B}, \mu) \qquad (6)$$

that will be called *De Rham complex* and *oriented De Rham complex* of the weighted network Γ, respectively. Moreover $H^n(\Gamma) = \ker \mathsf{d}_n / \mathrm{Img}\, \mathsf{d}_{n-1}$, $n = 0, 1$, will be called the *De Rham cohomology groups of* Γ, whereas $H^n_\tau(\Gamma) = \ker \mathsf{d}_{\tau_n} / \mathrm{Img}\, \mathsf{d}_{\tau_{n-1}}$, $n = 0, 1$, will be called the *oriented De Rham cohomology groups of* Γ. In addition, the *Betti numbers of* Γ are define as $\beta_n = \dim H^n(\Gamma)$, $n = 0, 1$.

As the constant are the unique functions whose gradient is null and the dimension of the space of irrotatial fields equals to $|E|$, we obtain the following well-known result.

Proposition 4.1 (Euler-Poincaré formula.) *If $(\Gamma, \mathsf{B}, \mu)$ is a weighted network and ν is weight on V, then $\beta_0 = 1$ and $\beta_1 = |E| - |V| + 1$. In particular, $\chi(\Gamma) = \beta_0 - \beta_1$.*

Corollary 4.2 *A weighted network is a tree iff $H^1(\Gamma)$ is trivial.*

It is clear that $\ker \mathsf{d}_{\tau_0} = \ker \mathsf{d}_0$ and $\dim \operatorname{Img} \mathsf{d}_{\tau_0} = \dim \operatorname{Img} \mathsf{d}_0$, since $\operatorname{Img} \mathsf{d}_{\tau_0} = \mathsf{D}_\tau(\operatorname{Img} \mathsf{d}_0)$. Therefore, $\beta_n = \dim H^n_\tau(\Gamma)$, $n = 0, 1$, and Proposition 4.1 and Corollary 4.2 are also true for oriented weighted networks. So, the results for the Betti numbers basically coincide with those obtained in [1, 7]. Moreover, the results here presented can be considered as an extension of the ones related with the orthogonal decomposition of a multigraph and its real cohomology obtained in [7].

We can also consider for $n = 0, 1$, δ_n and δ_{τ_n} the adjoint operators of d_n and d_{τ_n}, respectively. Therefore, we have the following dual complexes,

$$\mathbf{H}(\mathsf{B},\mu) \xrightarrow{\delta_1} \mathbf{H}(\mathsf{B},\mu) \xrightarrow{\delta_0} \mathsf{L}(\nu) \xrightarrow{\delta_{-1}} 0 \quad \text{and} \quad \mathbf{H}(\mathsf{B},\mu) \xrightarrow{\delta_{\tau_1}} \mathbf{H}(\mathsf{B},\mu) \xrightarrow{\delta_{\tau_0}} \mathbf{L}(\nu) \xrightarrow{\delta_{\tau_{-1}}} 0 \tag{7}$$

where $\delta_{-1} = \delta_{\tau_{-1}} = 0$, $\delta_0 = -\mathsf{div}_{\nu\mu}$, $\delta_1 = \mathsf{curl}^{\mathsf{B}}_\mu$, $\delta_{\tau_0} = -\mathsf{div}_{\tau\nu\mu}$ and $\delta_{\tau_1} = \mathsf{curl}^{\mathsf{B}}_{\tau\mu}$.

Following the guidelines of the differentiable case, we will call the endomorphisms determined by the identities

$$\delta_n \circ \mathsf{d}_n + \mathsf{d}_{n-1} \circ \delta_{n-1} \quad \text{and} \quad \delta_{\tau_n} \circ \mathsf{d}_{\tau_n} + \mathsf{d}_{\tau_{n-1}} \circ \delta_{\tau_{n-1}} \quad n = 0, 1,$$

Hodge Laplacian and *oriented Hodge Laplacian*, respectively. When $n = 0$, the Hodge and the oriented Hodge Laplacian are endomorphisms of $\mathsf{L}(\nu)$ that coincides with $-\Delta^{\mathsf{B}}_{\nu\mu}$ and $-\Delta^{\mathsf{B}}_{\tau\nu\mu}$, respectively. In addition, when $n = 1$, the Hodge and the oriented Hodge Laplacian are the endomorphisms of $\mathbf{H}(\mathsf{B},\mu)$ determined respectively by the identities

$$\boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} = \mathsf{curl}^{\mathsf{B}}_\mu \circ \mathsf{curl}^{\mathsf{B}}_\mu - \nabla^{\mathsf{B}} \circ \mathsf{div}_{\nu\mu} \quad \text{and} \quad \boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu} = \mathsf{curl}^{\mathsf{B}}_{\tau\mu} \circ \mathsf{curl}^{\mathsf{B}}_{\tau\mu} - \nabla^{\mathsf{B}}_\tau \circ \mathsf{div}_{\tau\nu\mu}. \tag{8}$$

Next, we analyze the fundamental properties of the Hodge and the oriented Hodge Laplacian on $\mathbf{H}(\mathsf{B},\mu)$.

Proposition 4.3 *The Hodge and the oriented Hodge Laplacians are self-adjoint positive semidefinite operators of second order and they satisfy the following identities*

$$\ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} = \{\mathsf{f} \in \mathbf{H}(\mathsf{B},\mu) : \mathsf{div}_{\nu\mu}\mathsf{f} = 0 \ \textit{and}\ \mathsf{curl}^{\mathsf{B}}_\mu \mathsf{f} = 0\}$$

$$\ker \boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu} = \{\mathsf{f} \in \mathbf{H}(\mathsf{B},\mu) : \mathsf{div}_{\tau\nu\mu}\mathsf{f} = 0 \ \textit{and}\ \mathsf{curl}^{\mathsf{B}}_{\tau\mu} \mathsf{f} = 0\}$$

Proof. As usual we only prove the unoriented case. The operator $\boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}$ is self-adjoint since $\mathsf{curl}^{\mathsf{B}}_\mu$ is a self-adjoint operator and ∇^{B} and $\mathsf{div}_{\nu\mu}$ are mutually adjoints. On the other hand, if $\mathsf{f} \in \mathbf{H}(\mathsf{B},\mu)$, then

$$\int_V \mathsf{B}(\boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}\mathsf{f}, \mathsf{f})\, \mu\, dx = \int_V \mathsf{B}(\mathsf{curl}^{\mathsf{B}}_\mu \mathsf{f}, \mathsf{curl}^{\mathsf{B}}_\mu \mathsf{f})\, \mu\, dx + 2\int_V (\mathsf{div}_{\nu\mu}\mathsf{f})^2 \nu\, dx,$$

which implies the claims. ∎

Proposition 4.4 (Hodge's Decomposition Theorem) *If* $(\Gamma, \mathsf{B}, \mu)$ *is a weighted network and* ν *is weight on* V*, the following orthogonal decompositions hold*

$$\mathbf{H}(\mathsf{B}, \mu) = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} \oplus \mathrm{Img}\, \nabla^{\mathsf{B}} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu} = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu} \oplus \mathrm{Img}\, \nabla^{\mathsf{B}}_{\tau} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu}.$$

Proof. Newly we only prove the unoriented case. It can be easily proved that the three subspaces are mutually orthogonal. In particular, this property implies that

$$\mathrm{Img}\, \nabla^{\mathsf{B}} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu} \subset \left(\ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}\right)^{\perp} = \mathrm{Img}\, \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} \subset \mathrm{Img}\, \nabla^{\mathsf{B}} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu},$$

since $\boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}$ is self-adjoint. Therefore it is verified that $\left(\ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}\right)^{\perp} = \mathrm{Img}\, \nabla^{\mathsf{B}} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu}$ and definitely

$$\mathbf{H}(\mathsf{B}, \mu) = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} \oplus \left(\ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}\right)^{\perp} = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} \oplus \mathrm{Img}\, \nabla^{\mathsf{B}} \oplus \mathrm{Img}\, \mathsf{curl}^{\mathsf{B}}_{\mu}. \quad \blacksquare$$

Corollary 4.5 *If* $(\Gamma, \mathsf{B}, \mu)$ *is a weighted network and* ν *is weight on* V*, then*

$$\ker \mathsf{curl}^{\mathsf{B}}_{\mu} = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu} \oplus \mathrm{Img}\, \nabla^{\mathsf{B}} \quad \textit{and} \quad \ker \mathsf{curl}^{\mathsf{B}}_{\tau} = \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu} \oplus \mathrm{Img}\, \nabla^{\mathsf{B}}_{\tau}.$$

Therefore, $H^1(\Gamma) \simeq \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}$*,* $H^1_\tau(\Gamma) \simeq \ker \boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu}$*. In particular,* $\boldsymbol{\Delta}^{\mathsf{B}}_{\nu\mu}$ *and* $\boldsymbol{\Delta}^{\mathsf{B}}_{\tau\nu\mu}$ *are automorphisms iff* Γ *is a tree.*

Keeping in mind the characterization of harmonic and oriented harmonic fields given in Proposition 4.3, we can re-interpreted the Hodge's Decomposition in terms of the discrete version of a well-known continuous equality, namely the *Helmholtz Theorem*: For each vector field f there exist vertex functions $u, \hat{u}$ and vector fields $\mathsf{g}, \mathsf{h}, \hat{\mathsf{g}}, \hat{\mathsf{h}}$, such that

$$\mathsf{f} = \nabla^{\mathsf{B}} u + \mathsf{curl}^{\mathsf{B}}_{\mu} \mathsf{g} + \mathsf{h} = \nabla^{\mathsf{B}}_{\tau} \hat{u} + \mathsf{curl}^{\mathsf{B}}_{\tau\mu} \hat{\mathsf{g}} + \hat{\mathsf{h}},$$

where $\mathsf{div}_{\nu\mu} \mathsf{h} = \mathsf{div}_{\tau\nu\mu} \hat{\mathsf{h}} = 0$ and $\mathsf{curl}^{\mathsf{B}}_{\mu} \mathsf{h} = \mathsf{curl}^{\mathsf{B}}_{\tau\mu} \hat{\mathsf{h}} = \mathsf{0}$. So the Hodge's Decomposition Theorem, or equivalently the Helmholtz Theorem, becomes not only an effective answer but also the accurate framework to the main question formulated in [14]. Specifically, there was raised the decomposition of a graph, or a purely resistive network, into a curl-free part, a divergence-free part and a simultaneously curl-free and divergence-free part.

5 Integral calculus on weighted networks

On this section we developed an integral calculus on weighted networks. We introduce the concept of circulation along a curve and we obtain a characterization of conservative fields. As one of the main objective of this work is to raise boundary value problems on networks we will pay special attention to the discrete version of Green's Identities as well as the Integration by Parts technique.

From now on we will restrict ourself to the unoriented case, since, similarly to the above sections, the oriented case only needs a few modification that can be easily carried out by the reader.

5.1 Integration along curves

One of the first question raised in continuous vector calculus is to define the line integration of a vector field, or equivalently the circulation of the field along a curve, whose physical meaning is the work necessary to carry a single particle from the curve's origen to its end. The question of if the realized work depends on or not depends on the followed trajectory, leads naturally to define the notion of conservative fields and raise the problem of how to characterize this property.

The discrete counterpart of this question should be useful in applications, specially in the framework of electrical networks, since, for instance, the well-known *cycle Kirchhoff's Law* could be accurately expressed. However, as far as the authors know this type of concepts have not been introduced precisely so far in the literature. Therefore, in this paragraph we will attempt to define them and to show their fundamental properties. Along the section we consider fixed a weighted network $(\Gamma, \mathsf{B}, \mu)$ an a weight, ν, on V. Moreover we denote by M the field of endomorphisms determined by B.

If $\alpha = \{x_j\}_{j=0}^n$ is a curve of length n on Γ, we will consider the vector field defined as $\mathsf{1}_\alpha(x) = \sum\limits_{\substack{x_j=x\\ j=0\dots,n-1}} e_{x_jx_{j+1}}$ if there exists $i = 0,\dots,n-1$ such that $x = x_i$ and $\mathsf{1}_\alpha(x) = \mathsf{0}$, otherwise. Moreover, the vector field $\mathsf{t}_\alpha = \dfrac{1}{\mu}\mathsf{1}_\alpha^{\mathsf{a}}$ is called *tangent field to the curve.* Clearly, $\mathsf{supp}(\mathsf{1}_\alpha) = \mathsf{supp}(\mathsf{t}_\alpha) = \alpha$, $\mathsf{div}_{\nu\mu}\mathsf{t}_\alpha = \frac{1}{\nu}\left(\varepsilon_{x_0} - \varepsilon_{x_n}\right)$ and hence t_α is a solenoidal vector field when α is a closed curve.

Given $\mathsf{f} \in \mathbf{H}(\mathsf{B},\mu)$ and α a curve on Γ, we define the *circulation of* f *along* α as the value

$$\int_\alpha \mathsf{f} = \frac{1}{2}\int_V \mathsf{B}(\mathsf{f},\mathsf{t}_\alpha)\,\mu\,dx. \tag{9}$$

If α is a curve with ends x_0 and x_n, the circulation of any gradient field along α can be calculate by applying (2). Specifically, we get that

$$\int_\alpha \nabla^{\mathsf{B}} u = u(x_n) - u(x_0), \qquad u \in \mathcal{C}(V)$$

and, in particular, the circulation of a gradient field along any closed curve equals 0. A vector field whose circulation along any closed curve is null is called *conservative.* So, any gradient field is conservative. Next, we characterize this kind of vector fields.

Proposition 5.1 *A vector field* f *is conservative iff there exists* $u \in \mathcal{C}(V)$ *verifying that* $\mathsf{d}u = (\mathsf{M}\mathsf{f})^{\mathsf{a}}$.

Proof. If we suppose that $\mathsf{d}u = (\mathsf{M}\mathsf{f})^{\mathsf{a}}$, then for any closed curve α we obtain that

$$\int_\alpha \mathsf{f} = \frac{1}{2}\int_V \langle \mathsf{M}\mathsf{f}, \mu\mathsf{t}_\alpha\rangle\,dx = \frac{1}{2}\int_V \langle (\mathsf{M}\mathsf{f})^{\mathsf{a}}, \mu\mathsf{t}_\alpha\rangle\,dx = \frac{1}{2}\int_V \mathsf{B}(\nabla^{\mathsf{B}} u, \mathsf{t}_\alpha)\,\mu\,dx = \int_\alpha \nabla^{\mathsf{B}} u = 0$$

and hence f is conservative.

Conversely, let $\mathsf{f} \in \mathbf{H}(\mathsf{B},\mu)$ be a conservative field and consider $x, y \in V$ and α_1, α_2 two curves from x to y, that is, $\alpha_1 = \{x_j\}_{j=0}^n$ and $\alpha_2 = \{y_j\}_{j=1}^m$, where $x_0 = y_0 = x$ and $x_n = x_m = y$. Then, the ordered sequence $\alpha_1 * \alpha_2 = \{z_j\}_{j=0}^{n+m}$ where $z_j = x_j$, $j = 0, \dots, n$ and $z_{n+j} = y_{m-j}$, $j = 1, \dots, m$ is a closed curve and verifies that $\mathsf{t}_{\alpha_1 * \alpha_2} = \mathsf{t}_{\alpha_1} - \mathsf{t}_{\alpha_2}$. Therefore $\displaystyle\int_{\alpha_1} \mathsf{f} = \int_{\alpha_2} \mathsf{f}$, since f is a conservative vector field.

Fixed $x \in V$, for any $y \in V$ there exists a curve from x to y, since Γ is connected. So, we can define unambiguously the function $u \in \mathcal{C}(V)$ as $u(x) = 0$ and $u(y) = \displaystyle\int_{\alpha_y} \mathsf{f}$ if $y \neq x$, where α_y is any curve from x to y. Moreover, if $z \sim y$ and α_{zy} denotes the curve $\{z, y\}$, then $\alpha_y * \alpha_{zy}$ is a curve from x to z, which implies that

$$u(z) - u(y) = \int_{\alpha_y * \alpha_{zy}} \mathsf{f} - \int_{\alpha_y} \mathsf{f} = -\int_{\alpha_{zy}} \mathsf{f}.$$

If $h \in \mathcal{C}(\Gamma)$ denotes de component function of Mf, as $\mathsf{t}_{\alpha_{zy}}(z) = \frac{1}{\mu(z)}\, e_{yz}$, $\mathsf{t}_{\alpha_{zy}}(y) = -\frac{1}{\mu(y)}\, e_{yz}$ and $\mathsf{t}_{\alpha_{zy}}(w) = \mathsf{0}$ otherwise, we obtain that

$$\int_{\alpha_{zy}} \mathsf{f} = \frac{1}{2}\int_V \langle \mathsf{Mf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle \, dx = \frac{1}{2}\langle \mathsf{Mf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle(z) + \frac{1}{2}\langle \mathsf{Mf}, \mu \mathsf{t}_{\alpha_{zy}} \rangle(y) = \frac{1}{2}\Big(h(z,y) - h(y,z)\Big)$$

and hence $\mathsf{d}u = (\mathsf{Mf})^{\mathsf{a}}$. ∎

Note that the above characterization of conservative vector fields can be re-written as follows: f is conservative iff there exists $u \in \mathcal{C}(V)$ such that $\mathsf{f} = \nabla^{\mathsf{B}} u + \mu \mathsf{M}^{-1} \mathsf{curl}_\mu^{\mathsf{B}} \mathsf{f}$. Therefore we obtain the following discrete counterpart of a well-known result in differentiable vector calculus.

Corollary 5.2 *An irrotational vector field is conservative iff it is a gradient field.*

5.2 Integration by Parts and Green's Identities

In this section we aim to establish the discrete analogous of the Integration by Parts technique, a central result on Riemannian manifolds. Moreover we are also interested in some useful consequences of it, namely the Divergence Theorem and the Green's Identities, that play a fundamental role in the analysis of boundary value problems. Naturally, these results are given on a vertex subset, so we need to define some concepts that represent the discrete analogous of the boundary and the exterior normal vector field to the set. Throughout this section we will consider ν and μ two fixed weights on V and we will also suppose that Γ is endowed with the canonical metric.

Given a vertex subset $F \subset V$, we will denote by F^c its complementary in V, by $\mathcal{C}(F)$ the set of real functions on V that vanish in F^c and by χ_F the characteristic function of F.

Moreover, we will call *interior of* F the set $\overset{\circ}{F}= \{x \in F : S_1(x) \subset F\}$, where $S_1(x)$ is the set of vertices that are adjacent to x, *boundary of* F the set $\delta(F) = \{x \in V : d(x, F) = 1\}$ and *closure of* F the set $\bar{F} = \{x \in V : d(x, F) \leq 1\} = F \cup \delta(F)$. In addition, when F is a proper subset, the *normal vector field to* F is defined as $\mathsf{n}_{_F} = -\mathsf{d}\chi_{_F}$.

Note that the component function of $\mathsf{n}_{_F}$ is given by $n_{_F}(x, y) = 1$ when $y \sim x$ and $(x, y) \in \delta(F^c) \times \delta(F)$, $n_{_F}(x, y) = -1$ when $y \sim x$ and $(x, y) \in \delta(F) \times \delta(F^c)$ and $n_{_F}(x, y) = 0$, otherwise. Therefore, $\mathsf{n}_{_{F^c}} = -\mathsf{n}_{_F}$, $\mathsf{supp}(\mathsf{n}_{_F}) = \delta(F^c) \cup \delta(F)$ and $\mathsf{n}_{_F}$ represents the exterior normal vector field to the subset.

Given a vector field f, for each $F \subset V$ non empty subset, we define the *restriction of* f *on* F as the vector field $\mathsf{f}_{_F}$ whose component function is given by $f_{_F} = f\chi_{_{\bar{F}\times\bar{F}\setminus\delta(F)\times\delta(F)}}$ where f is the component function of f. Observe that for any $x \in \delta(F)$, $\mathsf{f}_{_F}(x)$ has not *tangential nor exterior* components. Moreover $\mathsf{supp}(\mathsf{f}_{_F}), \mathsf{supp}(\mathsf{f}^s_{_F}), \mathsf{supp}(\mathsf{f}^a_{_F}) \subset \bar{F}$ and hence $\mathsf{supp}(\hat{\mathsf{f}}_{_F}) \subset \bar{F}$.

Proposition 5.3 (Integration by Parts) *Given* $F \subset V$ *a proper vertex subset, then for any vector field* $\mathsf{f} \in \mathcal{X}(\Gamma)$ *and functions* $u, v \in \mathsf{L}(\nu)$ *it is verified that*

$$\begin{aligned}\int_F v\langle \mathsf{f}, \mathsf{d}u\rangle\, \nu\, dx = &-\frac{1}{2}\int_{\bar{F}\times\bar{F}} (\nu f_{_F})^s(x, y)\,\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, dxdy\\ &- \int_{\bar{F}\times\bar{F}} (\nu f_{_F})^a(x, y)\, u(x)\, v(y)\, dxdy - \int_F (\mathsf{div}_{\nu\nu}\mathsf{f})\, uv\, \nu\, dx\\ &- \int_{\delta(F)} v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle\, \nu\, dx + \int_{\delta(F)} \langle (\nu\, \mathsf{f})^a, \mathsf{n}_{_F}\rangle\, uv\, dx\end{aligned}$$

and therefore,

$$\begin{aligned}\int_F \big(v\langle \mathsf{f}, \mathsf{d}u\rangle - u\langle \hat{\mathsf{f}}, \mathsf{d}v\rangle\big)\, \nu\, dx = &-2\int_F (\mathsf{div}_{\nu\nu}\mathsf{f})\, uv\, \nu\, dx\\ &+ \int_{\delta(F)} \big(u\langle \hat{\mathsf{f}}_{_F}, \mathsf{d}v\rangle - v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle\big)\, \nu\, dx + 2\int_{\delta(F)} \langle (\nu\, \mathsf{f})^a, \mathsf{n}_{_F}\rangle\, uv\, dx\end{aligned}$$

Proof. Firstly, we get that

$$\int_F v\langle \mathsf{f}, \mathsf{d}u\rangle\, \nu\, dx = \int_V v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle\, \nu\, dx - \int_{\delta(F)} v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle\, \nu\, dx.$$

Keeping in mind that $\mathsf{supp}(\mathsf{f}^s_{_F}), \mathsf{supp}(\mathsf{f}^a_{_F}) \subset \bar{F}$, applying Proposition 5.3 we obtain that

$$\begin{aligned}\int_V v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle\, \nu\, dx = &-\frac{1}{2}\int_{\bar{F}\times\bar{F}} (\nu f_{_F})^s(x, y)\,\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\, dxdy\\ &- \int_{\bar{F}} (\mathsf{div}_{\nu\nu}\mathsf{f}_{_F})\, uv\, \nu\, dx - \int_{\bar{F}\times\bar{F}} (\nu f_{_F})^a(x, y)\, u(x)\, v(y)\, dxdy,\end{aligned}$$

whereas applying Corollary 3.7 we get that

$$\int_V \big(v\langle \mathsf{f}_{_F}, \mathsf{d}u\rangle - u\langle \hat{\mathsf{f}}_{_F}, \mathsf{d}v\rangle\big)\, \nu\, dx = -2\int_{\bar{F}} (\mathsf{div}_{\nu\nu}\mathsf{f}_{_F})\, uv\, \nu\, dx.$$

The conclusions follow from the equalities $\mathsf{div}_{\nu\nu}\mathsf{f}_{_F} = \mathsf{div}_{\nu\nu}\mathsf{f}$ on F and $\nu\mathsf{div}_{\nu\nu}\mathsf{f}_{_F} = -\langle(\nu\mathsf{f})^{\mathsf{a}}, \mathsf{n}_{_F}\rangle$ on $\delta(F)$. ∎

Corollary 5.4 (Divergence Theorem) *Given $F \subset V$ a proper subset and $\mathsf{g} \in \mathcal{X}(\Gamma)$, it is verified that*

$$\int_F (\mathsf{div}_{\nu\mu}\,\mathsf{g})\,\nu\,dx = \int_{\delta(F)} \langle(\mu\,\mathsf{g})^{\mathsf{a}}, \mathsf{n}_{_F}\rangle\,dx.$$

Proof. The result follows taking $u = v = \chi_{\bar{F}}$ and $\mathsf{f} = \frac{\mu}{\nu}\,\mathsf{g}$ in the second identity of Proposition 5.3. ∎

Note that when $\mu = 1$ and $\nu = k$ the equality in the above corollary coincides with the one obtained in [20].

Our next objetive is to describe the discrete version of Green's Identities on F, for the second order operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$, where A is an arbitrary field of endomorphisms and F is a proper subset of V. For any $u \in \mathcal{C}(\bar{F})$, we get from (5) that

$$\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)(x) = \frac{1}{\nu(x)}\int_{\bar{F}} c^{\mathsf{A}}_{\mu}(x,y)\Big(u(x) - u(y)\Big)\,dy + q_{_\mathsf{A}}(x)u(x), \qquad x \in F, \tag{10}$$

where $q_{_\mathsf{A}}\colon F \longrightarrow \mathbb{R}$ is defined as

$$q_{_\mathsf{A}}(x) = \frac{1}{\nu(x)}\int_{\delta(\bar{F})} c^{\mathsf{A}}_{\mu}(x,y)\,dy = -\frac{1}{2\nu(x)}\int_{\delta(\bar{F})\times\delta(F)} \mu(z)a(z,x,y)\,dzdy, \qquad x \in F. \tag{11}$$

Note that $\mathsf{supp}(q_{_\mathsf{A}}) \subset \delta(F^c)$ and $q_{_\mathsf{A}} = 0$ when A is a diagonal field of endomorphisms. Moreover, from Lemma 3.10, we get that

$$\nu(x)q_{_\mathsf{A}}(x) = \frac{\mu(x)}{2}\langle \mathsf{A}\mathsf{1}, \mathsf{1}\rangle(x) + \frac{1}{2}\int_{\bar{F}} \mu(z)a(z,x,x)\,dz - \int_{\bar{F}} c^{\mathsf{A}}_{\mu}(x,y)\,dy, \qquad x \in F$$

and therefore,

$$q_{_{\mathsf{A}_\mathsf{a}}}(x) = \frac{1}{2}\Big(q_{_\mathsf{A}}(x) - q_{_{\mathsf{A}^t}}(x)\Big) = -\frac{1}{\nu(x)}\int_{\bar{F}} c^{\mathsf{A}_\mathsf{a}}_{\mu}(x,y)\,dy, \qquad x \in F. \tag{12}$$

The identity (10) shows that for any $u \in \mathcal{C}(\bar{F})$ the values of $\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)$ on F appears as the sum of two terms of different nature: The first one $\dfrac{1}{\nu(x)}\displaystyle\int_{\bar{F}} c^{\mathsf{A}}_{\mu}(x,y)\Big(u(x) - u(y)\Big)\,dy$, that we will called *the principal part of $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ on F*, looks like a combinatorial laplacian and depends on the connectivity between vertices in F as well as on the connectivity between vertices in F and in $\delta(F)$. The second one, $q_{_\mathsf{A}}u$, is a 0-order term that represents the kind of connectivity between F and its *exterior*, $(\bar{F})^c$. In other words, the operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ on $\mathcal{C}(\bar{F})$ is a *combinatorial Schrödinger operator* whose *ground state* is $q_{_\mathsf{A}}$, see [5]. We remark that when A is a diagonal field of endomorphism, the operator $\mathcal{L}^{\mathsf{A}}_{\nu\mu}$ is a first order operator that coincides with its principal part, but this is not true in the general case.

To develop a discrete version of Green's Identities it is also necessary to introduce a discrete analogue of the co-normal derivative for functions supported by $\bar{F}$. So, fixed F, for any field of endomorphisms A, we define the *co-normal derivative on F with respect to* A as the linear operator $\dfrac{\partial}{\partial \mathsf{n}_{\mathsf{A}}}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any $u \in \mathcal{C}(\bar{F})$ the function given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\right)(x) = \frac{1}{\nu(x)} \int_F c^{\mathsf{A}}_{\mu}(x,y) \Big(u(x) - u(y)\Big)\, dy, \qquad x \in \delta(F). \tag{13}$$

We define the *restriction of c^{A}_{μ} on F* as the function $b^{\mathsf{A}}_{\mu} = c^{\mathsf{A}}_{\mu}\, \chi_{\bar{F}\times\bar{F}\setminus\delta(F)\times\delta(F)}$ which keeps information about the connectivity between pairs of vertices in $F \times F$ or pairs of vertices in $F \times \delta(F)$, whereas the connectivity between vertices in $\delta(F)$ is neglected.

Proposition 5.5 (Green's Identities) *Given $F \subset V$ a proper vertex subset, then for any $u, v \in \mathcal{C}(\bar{F})$ it is verified that*

$$\begin{aligned}\int_F v\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\, dx &= \frac{1}{2}\int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_s}_{\mu}(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dxdy + \int_F q_{\mathsf{A}_s}\, uv\,\nu\, dx \\ &\quad + \int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_a}_{\mu}(x,y)\,u(x)\,v(y)\,dxdy + \int_{\delta(F)} h_{\mathsf{A}}\, uv\,\nu\,dx - \int_{\delta(F)} v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\,\nu\,dx\end{aligned}$$

where $h_{\mathsf{A}}\colon \delta(F) \longrightarrow \mathbb{R}$ is defined as $h_{\mathsf{A}}(x) = \dfrac{1}{\nu(x)}\displaystyle\int_F c^{\mathsf{A}_a}_{\mu}(x,y)\,dy$. Therefore,

$$\int_F \Big(v\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u) - u\mathcal{L}^{\mathsf{A}^t}_{\nu\mu}(v)\Big)\,\nu\,dx = \int_{\delta(F)}\left(u\frac{\partial v}{\partial \mathsf{n}_{\mathsf{A}^t}} - v\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\right)\nu\,dx + 2\int_{\delta(F)} h_{\mathsf{A}}\,uv\,\nu\,dx.$$

Proof. Applying the expression (10), we obtain that for any $u, v \in \mathcal{C}(\bar{F})$

$$\begin{aligned}\int_F v\,\mathcal{L}^{\mathsf{A}}_{\nu\mu}(u)\,\nu\,dx &= \int_{F\times\bar{F}} c^{\mathsf{A}}_{\mu}(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx + \int_F q_{\mathsf{A}} uv\nu\,dx \\ &= \int_{\bar{F}\times\bar{F}} b^{\mathsf{A}}_{\mu}(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx + \int_F q_{\mathsf{A}} uv\nu\,dx - \int_{\delta(F)} v\,\frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}\,\nu\,dx.\end{aligned}$$

On the other hand, reasoning as in the proof of Corollary 3.11, we get that

$$\begin{aligned}\int_{\bar{F}\times\bar{F}} b^{\mathsf{A}}_{\mu}(x,y)\Big(u(x)-u(y)\Big)v(x)\,dydx &= \frac{1}{2}\int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_s}_{\mu}(x,y)\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dydx \\ &\quad + \int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_a}_{\mu}(x,y)\,u(x)\,v(y)\,dxdy \\ &\quad - \int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_a}_{\mu}(x,y)\,u(y)\,v(y)\,dxdy.\end{aligned}$$

Moreover, tacking into account the identity (12) we obtain that

$$\begin{aligned}\int_{\bar F\times\bar F} b_\mu^{\mathsf{A}_\mathsf{a}}(x,y)\,u(y)\,v(y)\,dxdy &= \int_F u(y)\,v(y)\left(\int_{\bar F} b_\mu^{\mathsf{A}_\mathsf{a}}(x,y)\,dx\right)dy\\ &\quad+\int_{\delta(F)} u(y)\,v(y)\left(\int_F b_\mu^{\mathsf{A}_\mathsf{a}}(x,y)\,dx\right)\,dy\\ &= \int_F q_{\mathsf{A}_\mathsf{a}} uv\,\nu\,dy-\int_{\delta(F)} h_{\mathsf{A}} uv\,\nu\,dy\end{aligned}$$

and hence, the first Green's identity follows. Finally, keeping in mind that $b_\mu^{\mathsf{A}^t_\mathsf{a}}=-b_\mu^{\mathsf{A}_\mathsf{a}}$ and that $h_{\mathsf{A}^t}=-h_{\mathsf{A}}$, from the identity

$$\int_{\bar F\times\bar F} b_\mu^{\mathsf{A}_\mathsf{a}}(x,y)u(x)\,v(y)\,dxdy=\int_{\bar F\times\bar F} b_\mu^{\mathsf{A}^t_\mathsf{a}}(x,y)\,v(x)\,u(y)\,dxdy,$$

the Second Green's Identity follows subtracting to the First Green's Identity for $\mathcal{L}_{\nu\mu}^{\mathsf{A}}$ the corresponding one for $\mathcal{L}_{\nu\mu}^{\mathsf{A}^t}$. ∎

When $\mu=1$ and A is a diagonal field, the above Green's Identities correspond to those obtained by several authors, see principally [3, 13, 18].

6 Boundary Value Problems on weighted networks

Our aim in this section is to describe boundary value problems on a subset of a weighted network associated to second order operators on $\mathcal{C}(V)$ as well as to provide its variational or weak formulation. From now on suppose fixed the canonical metric, ν and μ weights on V. We also consider fixed a field of endomorphisms A, a vector field f, a vertex function q, and the endomorphism of $\mathsf{L}(\nu)$ given by

$$\mathcal{L}(u)=-\mathsf{div}_{\nu\mu}(\mathsf{A}\mathsf{d}u)+\langle\mathsf{f},\mathsf{d}u\rangle+q\,u. \tag{14}$$

Fixed $F\subset V$ a non empty connected subset, we consider $h\in\mathcal{C}(\delta(F))$ and the boundary operator $\mathcal{U}:\mathcal{C}(\bar F)\longrightarrow\mathcal{C}(\delta(F))$ defined as

$$\mathcal{U}(u)=\frac{\partial u}{\partial\mathsf{n}_{\mathsf{A}}}+\langle\mathsf{f}_F,\mathsf{d}u\rangle+h\,u. \tag{15}$$

Since $\langle\mathsf{f},\mathsf{d}u\rangle=\dfrac{1}{\nu}\langle\nu\mathsf{f},\mathsf{d}u\rangle=\dfrac{1}{\nu}\langle(\nu\mathsf{f})^s,\mathsf{d}u\rangle+\dfrac{1}{\nu}\langle(\nu\mathsf{f})^a,\mathsf{d}u\rangle$ and $\dfrac{1}{\nu}\langle(\nu\mathsf{f})^s,\mathsf{d}u\rangle=\mathsf{div}_{\nu\mu}(\mathsf{D}_\mathsf{g}\mathsf{d}u)$ where $\mathsf{g}=\frac{1}{\mu}(\nu\mathsf{f})^s$, by taking $\hat{\mathsf{A}}=\mathsf{A}-\mathsf{D}_\mathsf{g}$, we can re-write the above operators as

$$\mathcal{L}(u)=-\mathsf{div}_{\nu\mu}(\hat{\mathsf{A}}\mathsf{d}u)+\frac{1}{\nu}\langle(\nu\mathsf{f})^a,\mathsf{d}u\rangle+q\,u\quad\text{and}\quad\mathcal{U}(u)=\frac{\partial u}{\partial\mathsf{n}_{\hat{\mathsf{A}}}}+\frac{1}{\nu}\langle(\nu\mathsf{f}_F)^a,\mathsf{d}u\rangle+h\,u,$$

respectively. Therefore, without loss of generality, in the sequel we will suppose that the vector field f satisfies that $\mathsf{f} \in \frac{1}{\nu}\mathcal{X}^a(\Gamma)$. In particular this property implies that $\hat{\mathsf{f}} = -\mathsf{f}$.

Given $\delta(F) = H_1 \cup H_2$ a partition of $\delta(F)$ and functions $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$, a *boundary value problem on* F consists on finding $u \in \mathcal{C}(\bar{F})$ such that

$$\begin{aligned} -\mathsf{div}_{\nu\mu}(\mathsf{A}\mathsf{d}u)(x) + \langle \mathsf{f}, \mathsf{d}u\rangle(x) + q(x)\, u(x) &= g(x), \quad x \in F, \\ \frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}}(x) + \langle \mathsf{f}_F, \mathsf{d}u\rangle(x) + h(x)\, u(x) &= g_1(x), \quad x \in H_1, \\ u(x) &= g_2(x), \quad x \in H_2. \end{aligned} \tag{16}$$

In addition, *the associated homogeneous boundary value problem* consists on finding $u \in \mathcal{C}(\bar{F})$ such that $\mathcal{L}(u) = 0$ on F, $\mathcal{U}(u) = 0$ on H_1 and $u = 0$ on H_2. It is clear that the set of solutions of the homogeneous boundary value problem is a vector subspace of $\mathcal{C}(F \cup H_1)$ that we will denote by $\mathcal{V}_{F,H_1}$. Moreover if Problem (16) has solutions and u is a particular one, then $u + \mathcal{V}_{F,H_1}$ describes the set of all its solutions.

Problem (16) is generically known as a *mixed Dirichlet-Robin problem*, specially when $h \neq 0$, and $H_1, H_2 \neq \emptyset$, and summarizes the different boundary value problems that appear in the literature with the following proper names:

i) *Poisson equation*: $H_1 = H_2 = \emptyset$ and therefore $F = V$.

ii) *Dirichlet problem*: F a proper subset, $H_1 = \emptyset$ and therefore $H_2 = \delta(F)$.

iii) *Robin problem*: F a proper subset, $h \neq 0$, $H_2 = \emptyset$ and therefore $H_1 = \delta(F)$.

iv) *Neumann problem*: F a proper subset, $h = 0$, $H_2 = \emptyset$ and therefore $H_1 = \delta(F)$.

v) *Mixed Dirichlet-Neumann problem*: $h = 0$ and $H_1, H_2 \neq \emptyset$.

Consider now the endomorphism of $\mathsf{L}(\nu)$ given by

$$\mathcal{L}^*(u) = -\mathsf{div}_{\nu\mu}(\mathsf{A}^t\mathsf{d}u) - \langle \mathsf{f}, \mathsf{d}u\rangle + (q - 2\,\mathsf{div}_{\nu\nu}\mathsf{f})u, \tag{17}$$

the boundary operator on $\mathcal{C}(\delta(F))$

$$\mathcal{U}^*(u) = \frac{\partial u}{\partial \mathsf{n}_{\mathsf{A}^t}} - \langle \mathsf{f}_F, \mathsf{d}u\rangle + \Big(h + 2h_{\mathsf{A}} + 2\langle \mathsf{f}, \mathsf{n}_F\rangle\Big)\, u \tag{18}$$

and the (homogeneous) boundary value problem on F

$$\mathcal{L}^*(u) = 0 \ \text{ on } \ F, \quad \mathcal{U}^*(u) = 0 \ \text{ on } \ H_1 \quad \text{ and } \quad u = 0 \ \text{ on } \ H_2. \tag{19}$$

The above problem is called *the adjoint of (16)* and the subspace of its solutions will be denoted by $\mathcal{V}^*_{F,H_1}$. Moreover, we say that Problem (16) is *self-adjoint* when $\mathcal{L} = \mathcal{L}^*$ on F and $\mathcal{U} = \mathcal{U}^*$ on H_1. This property implies that $\mathsf{div}_{\nu\nu}\mathsf{f} = 0$ on F and that $h_{\mathsf{A}} + \langle \mathsf{f}, \mathsf{n}_F\rangle = 0$ on

H_1. In particular problem (16) is self-adjoint when A is a symmetric field of endomorphisms and moreover $\mathsf{f} = \mathsf{0}$ on $\bar{F}$ (observe that when A is symmetric then $h_{\mathsf{A}} = 0$).

To describe the conditions that assure the existence and uniqueness of solutions of the boundary value problem (16) we need to extend the Second's Green Identity to operators $\mathcal{L}$ and $\mathcal{L}^*$.

Proposition 6.1 *For any $u, v \in \mathcal{C}(\bar{F})$ it is verified that*

$$\int_F \Big(v\mathcal{L}(u) - u\mathcal{L}^*(v)\Big)\,\nu\, dx = \int_{\delta(F)} \Big(u\mathcal{U}^*(v) - v\mathcal{U}(u)\Big)\,\nu\, dx.$$

In particular, problems (16) and (19) are mutually adjoint.

Proof. As $\mathcal{L}(u) = \mathcal{L}^{\mathsf{A}}_{\nu\mu}(u) + \langle \mathsf{f}, \mathsf{d}u\rangle + qu$ and $\mathcal{L}^*(u) = \mathcal{L}^{\mathsf{A}^t}_{\nu\mu}(u) - \langle \mathsf{f}, \mathsf{d}u\rangle + (q - 2\,\mathsf{div}_{\nu\nu}\mathsf{f})\,u$, the first claim is a direct consequence of the Second's Green Identity and the second equality in the Integration by Parts proposition.

Consider now $u, v \in \mathcal{C}(F \cup H_1)$ such that $\mathcal{U}(u) = \mathcal{U}^*(v) = 0$ on H_1. Then

$$\int_{\delta(F)} \Big(u\mathcal{U}^*(v) - v\mathcal{U}(u)\Big)\,\nu\, dx = 0$$

and hence $\displaystyle\int_F v\,\mathcal{L}(u)\,\nu\, dx = \int_F u\,\mathcal{L}^*(v)\,\nu\, dx$, that is problems (16) and (19) are mutually adjoint. ∎

Proposition 6.2 (Fredholm Alternative) *Given $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$, the boundary value problem*

$$\mathcal{L}(u) = g, \quad on \;\; F, \qquad \mathcal{U}(u) = g_1 \;\; on \;\; H_1 \;\; and \;\; u = g_2 \;\; on \;\; H_2$$

has solution iff

$$\int_F gv\,\nu\, dx + \int_{H_1} g_1 v\,\nu\, dx + \int_{H_2} g_2 \langle \mathsf{f}_{_F}, \mathsf{d}v\rangle\,\nu\, dx = \int_{H_2} g_2 \frac{\partial v}{\partial \mathsf{n}_{\mathsf{A}^t}}\,\nu\, dx, \qquad for\ each\ v \in \mathcal{V}^*_{F,H_1}.$$

In addition, when the above condition holds, then there exists a unique solution of the boundary value problem in $(\mathcal{V}^*_{F,H_1})^\perp$, i.e. a unique solution u, such that

$$\int_{\bar{F}} uv\,\nu\, dx = 0, \quad for\ any\ v \in \mathcal{V}^*_{F,H_1}.$$

Proof. First observe that problem (16) is equivalent to the boundary value problem

$$\mathcal{L}(u) = g - \mathcal{L}(g_2), \quad \text{on} \;\; F, \;\; \mathcal{U}(u) = g_1 - \mathcal{U}(g_2), \quad \text{on} \;\; H_1, \quad \text{and} \;\; u = 0, \quad \text{on} \;\; H_2$$

in the sense that u is a solution of the this problem iff $u+g_2$ is a solution of (16).

Consider now the linear operators $\mathcal{F},\mathcal{F}^*\colon\mathcal{C}(F\cup H_1)\longrightarrow\mathcal{C}(F\cup H_1)$ defined as

$$\mathcal{F}(u)=\begin{cases}\mathcal{L}(u), & \text{on } F,\\ \mathcal{U}(u), & \text{on } H_1,\end{cases}\qquad\text{and}\qquad \mathcal{F}^*(u)=\begin{cases}\mathcal{L}^*(u), & \text{on } F,\\ \mathcal{U}^*(u), & \text{on } H_1,\end{cases}$$

respectively. Then, by applying Proposition 6.1 for any $u,v\in\mathcal{C}(F\cup H_1)$ it is verified that

$$\begin{aligned}\int_{F\cup H_1} v\mathcal{F}(u)\,\nu\,dx &= \int_F v\mathcal{L}(u)\,\nu\,dx+\int_{\delta(F)} v\mathcal{U}(u)\,\nu\,dx\\ &= \int_F u\mathcal{L}^*(v)\,\nu\,dx+\int_{\delta(F)} u\mathcal{U}^*(v)\,\nu\,dx=\int_{F\cup H_1} u\mathcal{F}^*(v)\,\nu\,dx.\end{aligned}$$

Therefore the operators $\mathcal{F}$ and $\mathcal{F}^*$ are mutually adjoint with respect to the inner product induced in $\mathcal{C}(F\cup H_1)$ by the weight ν. By applying the classical Fredholm Alternative we obtain that $\mathrm{Img}\mathcal{F}=(\ker\mathcal{F}^*)^\perp$. Clearly, the subspace $\ker\mathcal{F}^*$ coincides with the space of solutions of the homogeneous problem (19) and moreover problem (16) has a solution iff the function $\tilde g\in\mathcal{C}(F\cup H_1)$ given by $\tilde g=g-\mathcal{L}(g_2)$ on F and $g=g_1-\mathcal{U}(g_2)$ on H_1 verifies that $\tilde g\in\mathrm{Img}\mathcal{F}$. Therefore, problem (16) has solution iff for any $v\in\mathcal{V}^*_{F,H_1}$

$$\begin{aligned}0=\int_{F\cup H_1}\tilde g v\,\nu\,dx &= \int_F gv\,\nu\,dx+\int_{H_1} g_1 v\,\nu\,dx-\int_F v\mathcal{L}(g_2)\,\nu\,dx-\int_{H_1} v\mathcal{U}(g_2)\,\nu\,dx\\ &= \int_F gv\,\nu\,dx+\int_{H_1} g_1 v\,\nu\,dx-\int_F g_2\mathcal{L}^*(v)\,\nu\,dx-\int_{\delta(F)} g_2\mathcal{U}^*(v)\,\nu\,dx\\ &= \int_F gv\,\nu\,dx+\int_{H_1} g_1 v\,\nu\,dx-\int_{H_2} g_2\mathcal{U}^*(v)\,\nu\,dx.\end{aligned}$$

The result follows keeping in mind that $\mathcal{U}^*(v)=\dfrac{\partial v}{\partial \mathsf{n}_{\mathsf{A}^t}}-\langle \mathsf{f}_F,\mathsf{d}v\rangle$ on H_2, since $v=0$ on H_2. Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition is attained there exists a unique $w\in(\ker\mathcal{F}^*)^\perp$ such that $\mathcal{F}(w)=\tilde g$. Therefore, $u=w+g_2$ is the unique solution of problem (16) such that for any $v\in\ker\mathcal{F}^*$ verifies

$$\int_{\bar F} uv\,\nu\,dx=\int_{F\cup H_1} uv\,\nu\,dx=\int_{F\cup H_1} wv\,\nu\,dx=0,$$

since $v=0$ on H_2 and $g_2=0$ on $F\cup H_1$. ∎

Observe that as a by-product of the above proof, we obtain that $\dim\mathcal{V}_{F,H_1}=\dim\mathcal{V}^*_{F,H_1}$ and then we can conclude that uniqueness is equivalent to existence for any data.

Next, we establish the variational formulation of the boundary value problem (16), that will represent the discrete version of the weak formulation for boundary value problems. In particular, we will show that given $\mathsf{A}\in\mathcal{T}^1(\Gamma)$, $\mathsf{f}\in\frac{1}{\nu}\mathcal{X}^a(\Gamma)$ and $q\in\mathcal{C}(V)$, then for each non empty and connected vertex subset, F, the boundary operators *naturally* associated with the operator $\mathcal{L}(u) = \mathcal{L}^{\mathsf{A}}_{\nu\mu}(u) + \langle \mathsf{f}, \mathsf{d}u\rangle + q\,u$ are precisely those of the form $\mathcal{U}(u) = \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{A}}} + \langle \mathsf{f}_{F}, \mathsf{d}u\rangle + h\,u$.

Prior to describe the claimed formulation, we give some useful definitions. We define *the bilinear form* associated with the boundary value problem (16) as $\mathcal{B}_{\mathcal{L},\mathcal{U}}: \mathcal{C}(\bar{F}) \times \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by

$$\begin{aligned}\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = &\frac{1}{2}\int_{\bar{F}\times\bar{F}} b^{\mathsf{A}_{\mathsf{s}}}_{\mu}(x,y)\,\Big(u(x)-u(y)\Big)\Big(v(x)-v(y)\Big)\,dxdy\\ &+\int_{\bar{F}\times\bar{F}} \Big(b^{\mathsf{A}_{\mathsf{a}}}_{\mu}(x,y) - \nu(x) f_{F}(x,y)\Big)\,u(x)v(y)\,dxdy\\ &+\int_{F}(q + q_{\mathsf{A}_{\mathsf{s}}} - \mathsf{div}_{\nu\nu}\mathsf{f})\,uv\,\nu\,dx + \int_{\delta(F)}\Big(h + h_{\mathsf{A}} + \langle \mathsf{f}, \mathsf{n}_{F}\rangle\Big)uv\,\nu\,dx.\end{aligned} \tag{20}$$

Applying the Green's Identities and the Integration by Parts formulae, we obtain that

$$\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = \int_{F} v\mathcal{L}(u)\,\nu\,dx + \int_{\delta(F)} v\mathcal{U}(u)\,\nu\,dx, \quad \text{for any } u,v\in\mathcal{C}(\bar{F}) \tag{21}$$

and hence, from Proposition 6.1, we obtain that $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = \mathcal{B}_{\mathcal{L}^*,\mathcal{U}^*}(v,u)$ for any $u,v\in\mathcal{C}(\bar{F})$. Therefore, Problem (16) is self-adjoint iff $\mathsf{div}_{\nu\nu}\mathsf{f} = 0$ on F and $b^{\mathsf{A}_{\mathsf{a}}}_{\mu}(x,y) = \nu(x) f_{F(x,y)}$ for any $(x,y)\in\bar{F}\times\bar{F}$, that is iff $c^{\mathsf{A}_{\mathsf{a}}}_{\mu}(x,y) = \nu(x)f(x,y)$ for any $(x,y)\in\bar{F}\times\bar{F}\setminus\delta(F)\times\delta(F)$. Observe that the last equality implies that $c^{\mathsf{A}_{\mathsf{a}}}_{\mu}(x,y) = 0$ for $(x,y)\in\bar{F}\times\bar{F}\setminus\delta(F)\times\delta(F)$ such that $d(x,y)\geq 2$ and $h_{\mathsf{A}} + \langle \mathsf{f}, \mathsf{n}_{F}\rangle = 0$ on $\delta(F)$.

Associated with any pair of functions $g\in\mathcal{C}(F)$ and $g_1\in\mathcal{C}(H_1)$ we define the linear functional $\ell_{g,g_1}: \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ as $\ell_{g,g_1}(v) = \displaystyle\int_F gv\,\nu\,dx + \int_{H_1} g_1 v\,\nu\,dx$, whereas for any function $g_2\in\mathcal{C}(H_2)$ we consider the convex set $K_{g_2} = g_2 + \mathcal{C}(F\cup H_1)$.

Proposition 6.3 (Variational Formulation) *Given $g\in\mathcal{C}(F)$, $g_1\in\mathcal{C}(H_1)$ and $g_2\in\mathcal{C}(H_2)$, then $u\in K_{g_2}$ is a solution of Problem (16) iff*

$$\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = \ell_{g,g_1}(v), \quad \textit{for any } v\in\mathcal{C}(F\cup H_1)$$

and in this case, the set $u + \Big\{w\in\mathcal{C}(F\cup H_1) : \mathcal{B}_{\mathcal{L},\mathcal{U}}(w,v) = 0, \text{ for any } v\in\mathcal{C}(F\cup H_1)\Big\}$ describes all solutions of (16).

Proof. Since $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = \displaystyle\int_F v\mathcal{L}(u)\,\nu\,dx + \int_{\delta(F)} v\mathcal{U}(u)\,\nu\,dx$, for any $u,v\in\mathcal{C}(\bar{F})$, a function $u\in K_{g_2}$ satisfies that $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u,v) = \ell_{g,g_1}(v)$ for any $v\in\mathcal{C}(F\cup H_1)$ iff

$$\int_F v(\mathcal{L}(u) - g)\,\nu\,dx + \int_{H_1} v(\mathcal{U}(u) - g_1)\,\nu\,dx = 0.$$

Then, the first result follows by taking $v = \varepsilon_x$, $x \in F \cup H_1$. Finally, $u^* \in K_{g_2}$ is another solution of (16) iff $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u^*, v) = \ell_{g,g_1}(v)$ for any $v \in \mathcal{C}(F \cup H_1)$ and hence iff $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u - u^*, v) = 0$ for any $v \in \mathcal{C}(F \cup H_1)$. ∎

Observe that the equality $\mathcal{B}_{\mathcal{L},\mathcal{U}}(u, v) = \ell_{g,g_1}(v)$ for any $v \in \mathcal{C}(F \cup H_1)$ ensures that the condition of existence of solution given by the Fredholm Alternative is achieved, since for any $v \in \mathcal{C}(\bar{F})$ it is verified that

$$\int_F gv\,\nu\,dx + \int_{H_1} g_1 v\,\nu\,dx = \mathcal{B}_{\mathcal{L},\mathcal{U}}(u, v) = \mathcal{B}_{\mathcal{L}^*,\mathcal{U}^*}(v, u) = \int_F u\mathcal{L}^*(v)\,\nu\,dx + \int_{\delta(F)} u\mathcal{U}^*(v)\,\nu\,dx.$$

In particular if $v \in \mathcal{V}^*_{F,H_1}$ we get that

$$\int_F gv\,\nu\,dx + \int_{H_1} g_1 v\,\nu\,dx = \int_{H_2} u\mathcal{U}^*(v)\,\nu\,dx.$$

On the other hand, we note that the vector subspace

$$\Big\{w \in \mathcal{C}(F \cup H_1) : \mathcal{B}_{\mathcal{L},\mathcal{U}}(w, v) = 0, \text{ for any } v \in \mathcal{C}(F \cup H_1)\Big\}$$

is precisely the set of solutions of the homogeneous boundary value problem associated with (16). So, problem (16) has solution for any data g, g_1 and g_2 iff has a unique solution and this occurs iff $w = 0$ is the unique function in $\mathcal{C}(F \cup H_1)$ such that $\mathcal{B}_{\mathcal{L},\mathcal{U}}(w, v) = 0$, for any $v \in \mathcal{C}(F \cup H_1)$. Therefore, to assure the existence (and hence the uniqueness) of solutions of Problem (16) for any data it suffices to provide conditions under which $\mathcal{B}_{\mathcal{L},\mathcal{U}}(w, w) = 0$ with $w \in \mathcal{C}(F \cup H_1)$, implies that $w = 0$. To do this, we define $\mathcal{Q}_{\mathcal{L},\mathcal{U}}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ *the quadratic form* associated with the boundary value problem (16) as

$$\begin{aligned}\mathcal{Q}_{\mathcal{L},\mathcal{U}}(u) = \frac{1}{2}\int_{\bar{F}\times\bar{F}} b_\mu^{\mathsf{A}_s}(x, y)\Big(u(x) - u(y)\Big)^2 dxdy \\ + \int_F (q + q_{\mathsf{A}_s} - \mathsf{div}_{\nu\nu}\mathsf{f})\,u^2\nu\,dx + \int_{\delta(F)}\Big(h + h_{\mathsf{A}} + \langle \mathsf{f}, \mathsf{n}_F\rangle\Big)u^2\nu\,dx.\end{aligned} \tag{22}$$

Note that $\mathcal{Q}_{\mathcal{L},\mathcal{U}} = \mathcal{Q}_{\mathcal{L}^*,\mathcal{U}^*}$.

We will say that the pair (A, μ), where $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ is symmetric, is *strongly elliptic on* $\bar{F}$ if $b_\mu^{\mathsf{A}} \geq 0$ and $b_\mu^{\mathsf{A}}(x, y) > 0$ for any $(x, y) \in \bar{F} \times \bar{F} \setminus \delta(F) \times \delta(F)$, $x \sim y$.

Now we are ready to establish the two fundamental existence results.

Proposition 6.4 *Let $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ such that the pair (A_s, μ) is strongly elliptic on $\bar{F}$. Suppose that the functions q and h verify that $q \geq \mathsf{div}_{\nu\nu}\mathsf{f} - q_{\mathsf{A}_s}$ on F, $h \geq -h_{\mathsf{A}} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_1 and that it is not simultaneously satisfied that $q = \mathsf{div}_{\nu\nu}\mathsf{f} - q_{\mathsf{A}_s}$ on F, $h = -h_{\mathsf{A}} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_1 and $H_2 = \emptyset$. Then for any data $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$ the boundary value problem (16) has a unique solution.*

Proof. The hypotheses imply that $\mathcal{Q}_{\mathcal{L},\mathcal{U}}(v) \geq 0$ for any $v \in \mathcal{C}(F \cup H_1)$ and moreover $\mathcal{Q}_{\mathcal{L},\mathcal{U}}(v) = 0$ iff $(q + q_{\mathsf{A_s}} - \mathsf{div}_{\nu\nu}\mathsf{f})\, v^2 = 0$ on F, $\Big(h + h_{\mathsf{A}} + \langle \mathsf{f}, \mathsf{n}_F\rangle\Big)v^2 = 0$ on $\delta(F)$ and v is constant on $\bar{F}$. Therefore, when $H_2 \neq \emptyset$ necessarily $v = 0$ since v is null on H_2. On the other hand, when $H_2 = \emptyset$, newly v must be null unless that $q = \mathsf{div}_{\nu\nu}\mathsf{f} - q_{\mathsf{A_s}}$ on F and $h = -h_{\mathsf{A}} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on H_1, simultaneously. ∎

Proposition 6.5 *Let $\mathsf{A} \in \mathcal{T}^1(\Gamma)$ such that the pair (A_s, μ) is strongly elliptic on $\bar{F}$. Suppose that $H_2 = \emptyset$ and that $q = \mathsf{div}_{\nu\nu}\mathsf{f} - q_{\mathsf{A_s}}$ on F and $h = -h_{\mathsf{A}} - \langle \mathsf{f}, \mathsf{n}_F\rangle$ on $\delta(F)$. Then for any data $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(\delta(F))$, the boundary value problem (16) has solution iff it is verified that $\displaystyle\int_F g\,\nu\, dx + \int_{\delta(F)} g_1 \nu\, dx = 0$. Moreover, the solution is unique up to a constant and there exists a unique solution $u \in \bar{F}$ such that $\displaystyle\int_{\bar{F}} u\,\nu\, dx = 0$.*

Proof. The hypotheses imply that if $v \in \mathcal{C}(F \cup H_1)$ verifies $\mathcal{Q}_{\mathcal{L},\mathcal{U}}(v) = 0$ then v must be constant on $\bar{F}$. Therefore, the solutions of the adjoint boundary value problem are precisely the constant functions and hence from the Fredholm Alternative, problem (16) has solution iff $\displaystyle\int_F g\,\nu\, dx + \int_{\delta(F)} g_1 \nu\, dx = 0$. The rest of conclusions are also consequence of the Fredholm Alternative. ∎

When Problem (16) is self-adjoint and the pair (A_s, μ) is strongly elliptic, we can characterize the solutions of (16) by means of the discrete version of the celebrated *Dirichlet Principle.* Recall that when problem (16) is self-adjoint then its associated quadratic functional is given by

$$\mathcal{Q}_{\mathcal{L},\mathcal{U}}(u) = \frac{1}{2}\int_{\bar{F}\times\bar{F}} b_\mu^{\mathsf{A_s}}(x,y)\,\Big(u(x) - u(y)\Big)^2 dxdy + \int_F (q + q_{\mathsf{A_s}})\, u^2 \nu\, dx + \int_{\delta(F)} h u^2 \nu\, dx.$$

On the other hand, when the bilinear $\mathcal{B}_{\mathcal{L},\mathcal{U}}$ is symmetric and positive definite then the equality $\mathcal{Q}_{\mathcal{L},\mathcal{U}}(w) = 0$ is equivalent to the equality $\mathcal{B}_{\mathcal{L},\mathcal{U}}(w, v) = 0$, for any $v \in \mathcal{C}(F \cup H_1)$.

Corollary 6.6 (Dirichlet Principle) *Let $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$ and consider the quadratic functional $\mathcal{J}_{\mathcal{L},\mathcal{U}}\colon \mathcal{C}(\bar{F}) \longrightarrow \mathbb{R}$ given by*

$$\mathcal{J}_{\mathcal{L},\mathcal{U}}(u) = \mathcal{Q}_{\mathcal{L},\mathcal{U}}(u) - 2\ell_{g,g_1}(u).$$

If Problem (16) is self-adjoint, the pair (A_s, μ) is strongly elliptic, $q \geq -q_{\mathsf{A_s}}$ on F and $h \geq 0$ on H_1, then $u \in K_{g_2}$ is a solution of problem (16) iff it minimizes $\mathcal{J}_{\mathcal{L},\mathcal{U}}$ on K_{g_2}.

Proof. It suffices to note that the variational equality in the above proposition is in fact the *Euler identity* for the quadratic functional $\mathcal{Q}_{\mathcal{L},\mathcal{U}}$. ∎

We remark that when the pair (A_s,μ) is strongly elliptic then $q_{\mathsf{A}_s}\geq 0$. Therefore, under conditions of the Dirichlet Principle, the function q can take negative values on $\delta(F^c)$.

We conclude this paper with an application of the Dirichlet Principle to the problem of identification considered in [9], see also [6].

Proposition 6.7 *Let $\mathsf{A}_1,\mathsf{A}_2$ be two symmetric fields of endomorphisms such that the pairs (A_1,μ) and (A_2,μ) are strongly elliptic and it is verified that $b_\mu^{\mathsf{A}_2}\geq b_\mu^{\mathsf{A}_1}$. Consider also the functions $q_1,q_2\in\mathcal{C}(F)$ and $h_1,h_2\in\mathcal{C}(H_1)$ such that $q_2+q_{\mathsf{A}_2}\geq q_1+q_{\mathsf{A}_1}\geq 0$ and $h_2\geq h_1\geq 0$.*

Let $u_1,u_2\in\mathcal{C}(\bar F)$ such that $\mathcal{L}_{\nu\mu}^{\mathsf{A}_1}(u_1)+q_1u_1=\mathcal{L}_{\nu\mu}^{\mathsf{A}_2}(u_2)+q_2u_2=0$ on F, $\dfrac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}}+h_1u_1=\dfrac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}}+h_2u_2=0$ on H_1 and $u_1=u_2$, $\dfrac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}}=\dfrac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}}$ on H_2. Then, $u_1=u_2$ on $\bar F$, $q_2(x)+q_{\mathsf{A}_2}(x)=q_1(x)+q_{\mathsf{A}_1}(x)$ for any $x\in F$ such that $u_1(x)\neq 0$, $h_1(x)=h_2(x)$ for any $x\in H_1$ such that $u_1(x)\neq 0$ and moreover $b_\mu^{\mathsf{A}_2}(x,y)=b_\mu^{\mathsf{A}_1}(x,y)$ for any $x,y\in\bar F$ such that $u_1(x)\neq u_1(y)$.

Proof. If $g\in\mathcal{C}(H_2)$ is given by $g(x)=u_1(x)=u_2(x)$, $x\in H_2$, then u_1 and u_2 are respectively the unique solutions of the mixed Dirichlet-Robin boundary value problems

$$\mathcal{L}_{\nu\mu}^{\mathsf{A}_i}(u)+q_iu=0,\quad\text{on } F,\qquad \frac{\partial u_i}{\partial \mathsf{n}_{\mathsf{A}_i}}+h_iu_i=0,\quad \text{on } H_1\qquad\text{and}\qquad u_i=g,\quad\text{on } H_2\qquad i=1,2.$$

Therefore, if we consider the quadratic forms $\mathcal{Q}_1,\mathcal{Q}_2\colon\mathcal{C}(\bar F)\longrightarrow\mathbb{R}$ defined as

$$\mathcal{Q}_i(u)=\frac{1}{2}\int_{\bar F\times\bar F}b_\mu^{\mathsf{A}_i}(x,y)\Big(u(x)-u(y)\Big)^2dxdy+\int_F(q_i+q_{\mathsf{A}_i})\,u^2\nu\,dx+\int_{H_1}h_iu^2\nu\,dx,\quad i=1,2$$

then by applying the Dirichlet Principle, we know that u_i minimizes $\mathcal{Q}_i$ on K_g, $i=1,2$. Moreover, the hypotheses imply that $\mathcal{Q}_2(u)\geq\mathcal{Q}_1(u)$ for any $u\in K_g$. In addition, identity (21) implies that

$$\mathcal{Q}_1(u_1)=\int_{H_2}u_1\frac{\partial u_1}{\partial \mathsf{n}_{\mathsf{A}_1}}\,\nu\,dx=\int_{H_2}u_2\frac{\partial u_2}{\partial \mathsf{n}_{\mathsf{A}_2}}\,\nu\,dx=\mathcal{Q}_2(u_2)\geq\mathcal{Q}_1(u_2)$$

and hence $u_2=u_1$ on $\bar F$. Moreover, if $v=u_1=u_2$, then

$$\begin{aligned}0=&\frac{1}{2}\int_{\bar F\times\bar F}\Big(b_\mu^{\mathsf{A}_2}(x,y)-b_\mu^{\mathsf{A}_1}(x,y)\Big)\Big(v(x)-v(y)\Big)^2dxdy\\&+\int_F(q_2+q_{\mathsf{A}_2}-q_1-q_{\mathsf{A}_1})v^2\,\nu\,dx+\int_F(h_2-h_1)v^2\,\nu\,dx\end{aligned}$$

and the conclusions follow. ∎

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